QUANTUM SYMMETRIES STUDIED THROUGH THE LENS OF NON-LOCAL GAMES

A Dissertation

by

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ABSTRACT

Non-local games provide a useful framework for exhibiting the power of quantum entanglement, and we will focus our study on the graph isomorphism game and the metric isometry game. Work in quantum information theory has led to quantum versions of many concepts in classical mathematics, including quantum graphs and quantum metric spaces. We generalize Banica's construction of the quantum isometry group of a metric space to the class of quantum metric spaces in the sense of Kuperberg and Weaver.

We prove that the non-commutative algebraic notion of a quantum isomorphism between two finite, classical objects (either graphs or metric spaces) is the same as the more physically motivated one arising from the existence of a perfect quantum strategy for the corresponding game. This is achieved by showing that every algebraic quantum isomorphism between a pair of (quantum) objects X and Y arises from a certain measured bigalois extension for the quantum symmetry groups G_X and G_Y of X and Y. In particular, this implies that the quantum groups G_X and G_Y are monoidally equivalent.

For the case of the graph isomorphism game, we also establish a converse to this result, which says that a compact quantum group G is monoidally equivalent to the quantum automorphism group G_X of a given quantum graph X if and only if G is the quantum automorphism group of a quantum graph that is algebraically quantum isomorphic to X.

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1. INTRODUCTION AND BACKGROUND

1.1 Introduction

Finite input-output games have received considerable attention in the quantum information theory literature as tools for investigating the structure of *quantum correlations*. First proposed by physicist John Stewart Bell in the 1960's, a non-local game is played cooperatively by Alice and Bob against a referee; Alice and Bob may communicate prior to gameplay but are no longer able to communicate once a round begins. The two players may have access to a shared entangled quantum state and measurements performed on the entangled physical system allow them to correlate their answers to the referee in a way they would not be able to do classically ([13]). The quantum correlations are defined as the collection of conditional probability densities $(p(a, b|v, w))$ of obtaining a pair of outputs a, b for a given pair of inputs v, w. Their behaviour may be modeled by quantum mechanics and there are various different mathematical models (loc, q, qs, qc) describing the outcome of a quantum experiment. The conditional probability densities that can arise from quantum experiments in an entangled state is larger than the set of densities that can be obtained from classical shared randomness. Chapter 2 recalls definitions and results regarding non-local games and their classical and quantum strategies.

Work in quantum information theory has led to quantum versions of many concepts in classical mathematics. First introduced in [2] by taking two finite graphs X and Y, the graph isomorphism game $Iso(X, Y)$ has a winning classical strategy if and only if the two graphs are isomorphic. The game is played using two finite graphs X and Y, where the inputs and outputs are the disjoint union of the vertices from the two graphs.

A natural question to ask is whether such a game exists for other classical structures. We define a metric isometry game, $Isom(X, Y)$, for two finite metric spaces (X, d_X) and (Y, d_Y). This game has inputs and outputs that are the disjoint union of the set of points in X and Y and has a winning classical strategy if and only if the two metric spaces are isometric. The metric isometry game also has a close connection to the weighted graph isomorphism game, explained in Section 2.5.

Synchronous games form a special class of games where Alice and Bob share a set of questions and answers, and within a given round if both players receive the same question, they must produce the same answer. A bi-synchronous game has the additional restriction that the only way both players may win if they produce the same answer within a given round is if they were given the same question by the referee. The graph isomorphism game is an example of a bi-synchronous game that has been well-studied in the literature, and the metric isometry game is a new example in the class of bi-synchronous games.

Each synchronous game G has a $*$-algebra, $\mathcal{A}(G)$, that is associated to it and defined by generators and relations (Section 2.2). The representation theory of the game $*$-algebra gives information about the existence of perfect strategies for each of the mathematical models listed above ([24, 26]). For both the graph isomorphism game $G = Iso(X, Y)$ and the metric isometry game $G = Isom(X, Y)$, $\mathcal{A}(G)$ is a non-commutative analogue of the function algebra of the space of maps (either isomorphisms or isometries) $X \to Y$.

We say that the two objects X and Y are algebraically quantum isometric if $\mathcal{A}(G) \neq 0$. It was shown in [24] that there exist games G for which the $*$-algebra $\mathcal{A}(G)$ may be non-zero even if this algebra has no C^*-representations, and in particular, no perfect quantum strategies. We show that both the graph isomorphism game and the metric isometry game have the property that if the game $*$-algebra is nontrivial, then it has a nontrivial C^*-representation.

We are then naturally prompted to consider quantum analogues of classical objects such as quantum sets, quantum graphs, and quantum metric spaces.

We think of "quantum spaces" in the sense of non-commutative geometry: they are $*$-algebras or C*-algebras thought of as function algebras on the otherwise non-existent spaces. In the same spirit, we will work with *quantum graphs* (finite-dimensional C*-algebras equipped with some additional structure mimicking an "adjacency matrix") and *quantum groups* i.e. (objects dual to) non-commutative $*$-algebras with enough structure to resemble algebras of representative functions on compact groups, both are defined in Section 3.1.

As is the case classically, every quantum graph X has a *quantum automorphism group* G_X. The recent papers [30, 31, 32] uncover further remarkable connections between graph isomorphism games and quantum automorphism groups. Moreover, while [30] focuses on classical graphs, [31, 32] consider a more general categorical quantum mechanical framework which leads naturally to the notion of a quantum graphs and the generalization of the graph isomorphism game to that framework. In particular, [32] obtains a characterization of (finite-dimensional) quantum isomorphic quantum graphs X, Y in terms of simple dagger Frobenius monoids in the category of finite dimensional representations of the Hopf $*$-algebra $\mathcal{O}(G_X)$ of the corresponding quantum automorphism group G_X. On the other hand, [30] uses ideas from quantum group theory to establish the equivalence between the existence of C*-quantum isomorphisms for graphs and the existence of perfect strategies for the isomorphism game within the so-called *quantum commuting* framework.

Here, we continue in the same vein investigating connections between quantum automorphism groups of graphs and the graph isomorphism game, taking a somewhat dual approach to the one in [31, 32].

In [27], Kuperberg and Weaver define a non-commutative analogue of a metric space, called a W^*-quantum metric space, which we introduce in Section 3.2. A W^*-quantum metric space is a one-parameter family of weak*-closed operator systems $\mathcal{V} = \{\mathcal{V}_t\}_{t\geq 0}$. The intuition is that the $\mathcal{V}_t$ is a non-commutative analogue of pairs of points (x, y) whose distance is at most t.

Given a finite metric space with n points, we recall that the isometry group is a natural subgroup of the permutation group S_n. Specifically, the isometry group is the subgroup of S_n satisfying the relations $\sigma D = D\sigma$ where D is the distance matrix for the metric space and σ is a permutation in the symmetry group (viewed as a matrix group). In [5], Banica defined the quantum isometry group of a finite metric space in a similar way: the quantum isometry group is a quantum subgroup of the quantum permutation group, defined as the quotient of the function algebra of the quantum permutation group by adding relations mimicking the classical case. In Section 3.3, we generalize Bancia's definition to (possibly infinite) W^*-quantum metric spaces and show that the universal object defining the quantum isometry group exists in the finite dimensional case and agrees with Banica's definition.

One of our main results is proven in Chapter 4:

Theorem. *Consider two quantum objects (quantum graphs or quantum metric spaces), and suppose the quantum isomorphism / isometry space between the two quantum objects is non-zero. Then the two quantum symmetry groups corresponding to the two quantum objects are monoidally equivalent.*

If we look at this restricted to the classical case, we have the following result:

Theorem. *Two classical objects (graphs or metric spaces) are algebraically quantum isometric if and only if the corresponding game has a perfect quantum-commuting (qc)-strategy.*

This (non-commutative) bundle-theoretic perspective on $\mathcal{A}(G)$ has advantages: although the construction of $\mathcal{A}(G)$ is purely algebraic and does not assume the existence of any C*-representations of this object, we use the above result to show that this algebra always admits a faithful invariant state whenever it is non-zero, leading to connections with the notion of monoidal equivalence between quantum symmetry groups. Loosely speaking, we say that two compact quantum groups are *monoidally equivalent* if their categories of finite-dimensional unitary representations are equivalent as rigid C*-tensor categories.

One main result about the graph isomorphism game says:

Theorem. *Let X be a quantum graph and G_X its quantum automorphism group. Then for any compact quantum group G monoidally equivalent to G_X, one can construct from this monoidal equivalence a quantum graph Y, an isomorphism of quantum groups $G \cong G_Y$, and an algebraic quantum isomorphism $X \cong_{A^*} Y$.*

Remark. *The theorem above says that the collection of quantum groups $\{G_X\}$ where X is a quantum graph is closed under monoidal equivalence.*

The rest of this chapter contains a brief overview of compact quantum groups, which we will use to prove the results that follow. Chapter 2 recalls some preliminary material on non-local games and introduces the two examples we will study: the graph isomorphism game and the metric isometry game. In Chapter 3, we begin to look at quantum analogues of classical objects, namely quantum graphs and quantum metric spaces. We study the quantum symmetry groups of these quantum objects. We look at Galois extensions in the context of both these games in Chapter 4, and prove our main results in this chapter. Finally, Chapter 5 contains a summary of our results.

1.2 Notation

If n is a natural number, we sometimes write $[n]$ for the ordered set $\{1, 2, \ldots, n\}$. All vector spaces considered here are over the complex field. We use the standard leg numbering notation for linear operators on tensor products of vector spaces. For example, if X, Y, Z are vector spaces and $T : X \otimes Y \to X \otimes Y$ is a linear map, then $T_{13} : X \otimes Z \otimes Y \to X \otimes Z \otimes Y$ is the linear map which acts as T on the first and third leg of the triple tensor product, and as the identity on the second leg.

When referring to tensor products, we use the symbol $\otimes$ to denote the tensor product of Hilbert spaces or the minimal tensor product of C^*-algebras. We use the symbol $\overline{\otimes}$ to either denote the normal spatial tensor product of von Neumann algebras or the weak$*$ closure of the algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$.

1.3 Compact Quantum Groups

We will now review the basics of compact quantum groups, their actions and representations. The reader may be referred to references [46, 33, 45, 15] for details.

Definition 1.3.1. *A compact quantum group is a unital C^*-algebra $\mathcal{A}$ equipped with a unital $*$-homomorphism called comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that*

- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ *as homomorphisms (co-associativity)*
- *the spaces* $\mathrm{span}\{(a \otimes 1)\Delta(b) \mid a, b \in \mathcal{A}\}$ *and* $\mathrm{span}\{(1 \otimes a)\Delta(b) \mid a, b \in \mathcal{A}\}$ *are dense in* $\mathcal{A} \otimes \mathcal{A}$ *(the cancellation property)*

Motivation for this definition comes from the example given by $\mathcal{A} = C(G)$, the space of all continuous complex functions on a fixed compact group G. Here, comultiplication $\Delta : C(G) \to C(G \times G) \cong C(G) \otimes C(G)$ is given by $(\Delta(f))(g, h) = f(g \cdot h)$ so Δ captures the group operation at the level of $C(G)$.

Conversely, every compact quantum group $(\mathcal{A}, \Delta)$ whose underlying C^*-algebra $\mathcal{A}$ is commutative is of the form $\mathcal{A} = C(G)$ for some compact group G [46].

Remark 1.3.2. *Based on this commutative example, we use the notation $\mathcal{A} = C(G)$ for general compact quantum groups.*

We look at a few examples of compact quantum groups which will be used later. First, we define a *magic unitary* over a unital $*$-algebra $\mathcal{A}$ to be an $n \times n$ matrix $U = [u_{ij}]_{i,j}$ with entries $u_{ij} \in \mathcal{A}$ which satisfies

- $u_{ij} = u_{ij}^* = u_{ij}^2$
- $u_{ij}u_{i\ell} = \delta_{j\ell}u_{ij}$
- $u_{ij}u_{\ell j} = \delta_{i\ell}u_{ij}$
- $\sum_{i=1}^n u_{ij} = 1 = \sum_{j=1}^n u_{ij}$

In the case where $\mathcal{A}$ is the complex numbers, a magic unitary matrix is simply a permutation matrix.

Example 1.3.3. *The quantum permutation group S_n^+ [42] is the compact quantum group $(\mathcal{A}, \Delta)$ where $\mathcal{A} = C(S_n^+)$ is the universal C^*-algebra generated by the entires of an $n \times n$ magic unitary matrix $u = [u_{ij}]$. Comultiplication is given by the formula $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.*

If we were to instead consider the universal C^*-algebra generated by commuting entries of an $n \times n$ magic unitary matrix, we would get the function algebra $C(S_n)$ of the symmetry group S_n. Thus, we should view $C(S_n^+)$ as a non-commutative symmetry group of a finite set of n points with *no extra structure. There always exists a quotient map π from $C(S_n^+)$ into $C(S_n)$ which intertwines the coproducts on $C(S_n^+)$ and $C(S_n)$; that is, we can realize S_n as a subgroup of S_n^+.*

It was shown that for $n \geq 4$, Wang showed in [42] that $C(S_n^+)$ is non-commutative, that is, that even classical objects such as four points with no additional structure can have quantum symmetries unseen when restricting to classical groups.

Example 1.3.4. *The universal unitary quantum group U_F^+ associated to a matrix $F \in GL_n(\mathbb{C})$ [42] is the universal $*$-algebra generated by the entries of a $n \times n$ matrix $u = [u_{ij}]$ for which $(1 \otimes F)[u_{ij}^*](1 \otimes F^{-1})$ is a unitary in $M_n(C(U_F^+))$. The comultiplication map Δ is defined the same as for S_n^+.*

Let $G = (C(G), \Delta)$ be a compact quantum group and $\mathcal{H}$ a finite dimensional Hilbert space of dimension n. In general, a *representation of* G is an invertible element $v \in \mathcal{B}(\mathcal{H}) \otimes C(G)$ such that $(\mathrm{id} \otimes \Delta)(v) = v_{12}v_{13}$. If we fix an orthonormal basis (e_j) for $\mathcal{H}$, then a representation v corresponds to an invertible matrix $v = [v_{ij}] \in M_n(C(G))$ such that $\Delta(v) = \sum_{k=1}^n v_{ik} \otimes v_{kj}$. We call v a *unitary representation* if it is unitary. Given an infinite dimensional Hilbert space $\mathcal{H}$, one can similarly define an infinite dimensional unitary representation to be some $u \in M(K(\mathcal{H}) \otimes C(G))$ such that $(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}$. We refer the reader to [33] for details.

Fix two representations $v \in B(\mathcal{H}_v) \otimes C(G)$ and $u \in B(\mathcal{H}_u) \otimes C(G)$. A *morphism* between u and v is a linear map $T : \mathcal{H}_u \to \mathcal{H}_v$ that satisfies $(T \otimes 1)u = v(T \otimes 1)$, and we let $\mathrm{Mor}(u, v)$ be the Banach space of all morphisms between u and v. We call representations u and v *equivalent* if $\mathrm{Mor}(u, v)$ contains an invertible element. Two representations are said to be *irreducible* if $\mathrm{Mor}(u, u) = \mathbb{C}$. The set of equivalence classes of irreducible representations is denoted $\mathrm{Irr}(G)$. It's easy to show that if u is a unitary representation, then $\mathrm{Mor}(u, u)$ is a C^*-algebra. We may consider the *direct sum* $u \oplus v \in B(\mathcal{H}_u \oplus \mathcal{H}_v) \otimes C(G)$, the *tensor product* $u \otimes v := u_{13}v_{23} \in B(\mathcal{H}_u \otimes \mathcal{H}_v) \otimes C(G)$ and *conjugate representation* $\overline{u} := [u_{ij}^*] \in B(\overline{\mathcal{H}}) \otimes C(G)$.

We may view a compact quantum group as a Hopf $*$-algebra, with the the structure maps of the underlying compact group giving rise to a number of unital homomorphisms with Δ as above. More details may be found in [19].

Definition 1.3.5. *A Hopf $*$-algebra is a pair $(\mathcal{A}, \Delta)$ where $\mathcal{A}$ is a unital $*$-algebra and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a unital $*$-homomorphism that satisfies $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and for some $*$-homomorphism $\epsilon : \mathcal{A} \to \mathbb{C}$ satisfying $(\epsilon \otimes \mathrm{id})\Delta(a) = a = (\mathrm{id} \otimes \epsilon)\Delta(a)$ and for some anti-homomorphism $S : \mathcal{A} \to \mathcal{A}$ such that $m(S \otimes \mathrm{id})\Delta(a) = \epsilon(a)1 = m(\mathrm{id} \otimes S)\Delta(a)$ where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map.*

A special class of compact quantum groups are the compact matrix quantum groups. A *compact matrix quantum group* is a compact quantum group G with a finite dimensional representation $u = [u_{ij}]$ such that $C(G) = C^*(u_{ij}, u_{ij}^*)$. We call u a fundamental representation of G.

Most compact quantum groups are presented as compact matrix quantum groups: both examples above are examples of compact matrix quantum groups.

Definition 1.3.6. *Given a compact matrix quantum group G, we define the Hopf $*$-algebra $\mathcal{O}(G)$ to be the $*$-algebra generated by the coefficients u_{ij} of the fundamental representation $U = [u_{ij}]$.*

It is possible to define compact quantum groups by either Hopf $*$-algebras or defined as a C^*-algebraic object, and we will do this interchangeably throughout this dissertation.

Theorem 1.3.7. *For any compact quantum group G, the pair $(\mathcal{O}(G), \Delta)$ is a Hopf $*$-algebra with $S(u_{ij}) = u_{ji}^*$ and $\epsilon(u_{ij}) = \delta_{ij}$.*

The representation category of a compact quantum group G, denoted $\mathrm{Rep}(G)$, is defined to be the category whose objects are equivalence classes of finite dimensional representations of G and morphisms are intertwiners. An interested reader can refer to [33] for more details.

The fundamental theorem on finite dimensional representations of compact quantum groups is analogous to the classical case. It is stated as follows:

Theorem 1.3.8. *([46]) Let G be a compact quantum group. Every finite dimensional representation of G is equivalent to a unitary representation, and every finite dimensional unitary representation of G is equivalent to a direct sum of irreducible representations.*

$C(G)$ is densely linearly spanned by the matrix elements of irreducible unitary representations of G.

2. NON-LOCAL GAMES

2.1 Games and Correlations

A *two-player non-local game* is a tuple $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ where I_A, I_B, O_A, O_B are finite sets representing the inputs and outputs for Alice and Bob and

$$\lambda : I_A \times I_B \times O_A \times O_B \to \{0, 1\}$$

is a *rule function*. The game is played cooperatively by two players, Alice and Bob, against a referee. The game rules are known by all before the game begins, and Alice and Bob may agree on a strategy before beginning to play the game. While the game is being played however, Alice and Bob may no longer communicate and can only rely on the strategy they agreed upon.

A single round of the game consists of the referee giving Alice an input (question) v from her set of possible inputs I_A, and giving Bob an input w from his set of possible inputs I_B. Without communicating, Alice and Bob reply with outputs (answers) $a \in O_A$ and $b \in O_B$, respectively. They win the round if $\lambda(v, w, a, b) = 1$ and lose the round otherwise. Alice, Bob, and the referee play repeated rounds, and their goal is to win each round.

A game is called *synchronous* provided that the two players input sets are the same ($I = I_A = I_B$), as are their output sets ($O = O_A = O_B$), and the rule function satisfies $\lambda(v, v, a, b) = 0$ if $a \neq b$ for all $v \in I$.

Another way to say this is that when Alice and Bob receive the same input, in order to win they must produce the same output. We call a game *bi-synchronous* as in [34] provided that the game is both synchronous and $\lambda(v, w, a, a) = 0$ if $v \neq w$ for all $a \in O$.

The strategies that Alice and Bob may utilize break into two categories: either a deterministic strategy or a random strategy. A *deterministic strategy* is a pair of functions $h : I_A \to O_A$ and $k : I_B \to O_B$ which determine the answers Alice and Bob give to the referee. If they receive $(v, w) \in I_A \times I_B$ then they respond with $(h(v), k(w)) \in O_A \times O_B$. A deterministic strategy wins every round if and only if $\lambda(v, w, h(v), k(w)) = 1$ for all $(v, w) \in I_A \times I_B$. We call such a strategy a perfect deterministic strategy. Given a synchronous game, a *perfect deterministic strategy* must satisfy $h = k$.

A *random strategy* or *probabilistic strategy* is characterized by the fact that on different rounds of the game, Alice and Bob may produce different outputs given the same input pair (v, w). The idea is that even though there might not exist a perfect deterministic strategy to win the game, the players may improve their chance of winning the game by sampling their outputs according to some joint probability distribution. As an outsider to the game, one may observe multiple rounds of the game to obtain the conditional probability density $p(a, b|v, w)$ which describes their behaviours and represents the probability that given inputs $(v, w) \in I_A \times I_B$ that Alice and Bob produce outputs $(a, b) \in O_A \times O_B$. It's clear that $0 \leq p(a, b|v, w) \leq 1$ and that given a fixed $(v, w) \in I_A \times I_B$, $\sum_{a \in O_A, b \in O_B} p(a, b|v, w) = 1$.

We call a random strategy *perfect* if Alice and Bob win each round with probability 1. That is, the strategy is perfect if $\lambda(v, w, a, b) = 0$ implies that $p(a, b|v, w) = 0$ for any $(v, w, a, b) \in I_A \times I_B \times O_A \times O_B$.

Assuming different mathematical models, we may get different sets of conditional probabilities $p(a, b|v, w)$. Given n inputs and k outputs, we denote the set of conditional probability densities that belong to each of these sets by $C_t(n, k)$ satisfying

$$C_{loc}(n, k) \subseteq C_q(n, k) \subseteq C_{qs}(n, k) \subseteq C_{qa}(n, k) \subseteq C_{qc}(n, k) \subseteq C_{ns}(n, k)$$

where t is one of *local (loc)*, *quantum (q)*, *quantum spatial (qs)*, *quantum approximate (qa)*, *quantum commuting (qc)*, and *non-signalling (ns)* correspond to the different models. Here, local (or classical) correlations arise when Alice and Bob utilize only a shared probability space while quantum strategies arise from the random outcomes of entangled quantum experiments.

Definition 2.1.1. *The set $C_{loc}(n, k)$ is the set of classical conditional probabilities which arise from Alice and Bob sharing a probability space (Ω, μ) where each has a collection of random variables*

$f_{\omega,A} : I_A \to O_A$ and $g_{\omega,B} : I_B \to O_B$ with

$$p(a, b|v, w) = \mu(\{\omega \in \Omega \mid f_{\omega,A}(v) = a, g_{\omega,B}(w) = b\}).$$

It's known that the set $C_{loc}(n, k)$ is a closed, convex set for all n, k. Moreover, it is a polytope. We also have a number of different mathematical models to describe the quantum strategies that may arise, which we review below.

Definition 2.1.2. *A correlation $p(a, b|v, w)$ is called a quantum (q) correlation if Alice and Bob's state spaces are given by finite dimensional Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ on which they're allowed to make measurements, and a shared state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Alice has orthogonal projections $e_{v,a} \in B(\mathcal{H}_A)$ such that $\sum_a e_{v,a} = \mathrm{id}_{\mathcal{H}_A}$ (and Bob has orthogonal projections $f_{w,b} \in B(\mathcal{H}_B)$ such that $\sum_b f_{w,b} = \mathrm{id}_{\mathcal{H}_B}$) such that*

$$p(a, b|v, w) = \langle e_{v,a} \otimes f_{w,b}\psi, \psi\rangle.$$

Definition 2.1.3. *A correlation is called quantum spacial (qs) correlation if we relax the definition of quantum correlations to allow the Hilbert spaces to be infinite dimensional.*

Definition 2.1.4. *We let $C_{qa}(n, k)$ be the set of all quantum approximate (qa) correlations and define this to be the closure of $C_q(n, k)$.*

It's known in [39] that $C_{qa}(n, k) = \overline{C_{qs}(n, k)}$.

Definition 2.1.5. *A correlation $p(a, b|v, w)$ is called a quantum commuting (qc) correlation if Alice and Bob now share a single Hilbert space $\mathcal{H}$ on which their shared state ψ lives. It is now required that all of Alice's measurement operators $e_{v,a} \in B(\mathcal{H})$ commute with all of Bob's measurement operators $f_{w,b} \in B(\mathcal{H})$ and $p(a, b|v, w) = \langle e_{v,a} f_{w,b}\psi, \psi\rangle$.*

Definition 2.1.6. *We call a correlation $(p(a, b|v, w)$ non-signalling (ns) if the following definitions are well-defined for all $v \in I_A$ and $w \in I_B$:*

$$p_A(a|v) = \sum_{w \in O_B} p(a,b|v,w) = \sum_{w \in O_B} p(a,b|v,w') \qquad w, w' \in I_B$$

$$p_B(b|w) = \sum_{v \in O_A} p(a,b|v,w) = \sum_{v \in O_A} p(a,b|v',w) \qquad v, v' \in I_B$$

This implies that the two players do not communicate: so the conditional probability that Alice produces output $a \in O_A$ when she is given input $v \in I_A$ (or Bob produces output $b \in O_B$ when he is given input $w \in I_B$) is independent of the behaviour of the other player.

It's known that for $n, k \geq 2$ $C_{loc}(n,k) \neq C_q(n,k)$. It was shown in [20] that for $n \geq 5, k \geq 2$ we have $C_{qs}(n,k) \neq C_{qa}(n,k)$, and [16] showed for $n \geq 5, k \geq 3$ then $C_q(n,k) \neq C_{qs}(n,k)$. In [25] it was shown there exists n, k such that $C_{qa}(n,k) \neq C_{qc}(n,k)$ which also disproves Connes' embedding conjecture posed in [17]. We refer the reader to [26, 30] for a more thorough investigation of the models.

We say that a game has a *perfect t-strategy* if it has a perfect random strategy that belongs to one of these models, where t is one of *loc, q, qs, qa,* or *qc*.

2.2 The $*$-algebra of a synchronous game

In this subsection, we recall the definition of the $*$-algebra of a synchronous game and summarize the results found in [26, 30, 40].

Definition 2.2.1. *The $*$-algebra of a synchronous game $\mathcal{G}$, $\mathcal{A}(\mathcal{G})$, is defined as the quotient of the free $*$-algebra generated by $\{e_{v,a} \mid v \in I, a \in O\}$ subject to the relations*

- $e_{v,a} = e_{v,a}^*$
- $e_{v,a} = e_{v,a}^2$
- $1 = \sum_a e_{v,a}$

- $e_{v,a}e_{w,b} = 0$ *for all* v, w, a, b *such that* $\lambda(v, w, a, b) = 0$

The generators $e_{v,a}$ represent the measurement operators for Alice while the algebraic relations above are forced upon us by the restrictions of a winning strategy – from both the mathematical formalism of quantum mechanics together with the structure of the rule function. In particular, since our game is synchronous and moreover $\lambda(v, w, a, b) = \delta_{a,b}$ then if $a \neq b$ we have $e_{v,a}e_{v,b} = 0$. Note that this algebra may be zero, and in fact, we are specifically interested in the cases where this algebra is non-zero.

The following theorem proved in [26] shows that the representation theory of the game $*$-algebra is crucial to understanding the existence of a winning t-strategy of the game.

Theorem 2.2.2. *Let* $\mathcal{G} = (I, O, \lambda)$ *be a synchronous game. Then*

- $\mathcal{G}$ *has a perfect deterministic strategy if and only if* $\mathcal{G}$ *has a perfect loc-strategy if and only if there exists a unital* $*$*-homorphism from* $\mathcal{A}(\mathcal{G})$ *to* $\mathbb{C}$.

- $\mathcal{G}$ *has a perfect* q*-strategy if and only if* $\mathcal{G}$ *has a perfect* qs*-strategy if and only if there exists a unital* $*$*-homomorphism from* $\mathcal{A}(\mathcal{G})$ *to* $B(\mathcal{H})$ *for some non-zero finite dimensional Hilbert space* $\mathcal{H}$.

- $\mathcal{G}$ *has a perfect* qa*-strategy if and only if there exists a unital* $*$*-homomorphism of* $\mathcal{A}(\mathcal{G})$ *into the ultrapower of the hyperfinite* II_1*-factor.*

- $\mathcal{G}$ *has a perfect* qc*-strategy if and only if there exists a unital* C^**-algebra* $\mathcal{C}$ *with a faithful trace and a unital* $*$*-homomorphism* $\pi : \mathcal{A}(\mathcal{G}) \to \mathcal{C}$.

Definition 2.2.3. *We say a synchronous game* $\mathcal{G}$ *has a perfect* A^**-strategy if* $\mathcal{A}(\mathcal{G})$ *is non-zero. We say* $\mathcal{G}$ *has a perfect* C^**-strategy if there exists a unital* $*$*-homomorphism form* $\mathcal{A}(\mathcal{G})$ *into* $B(\mathcal{H})$ *for some non-zero Hilbert space* $\mathcal{H}$.

In general, these strategies are not physical and there is no guarantee of a corresponding physical correlation.

2.3 The Graph Isomorphism Game

A *graph* X is specified by a vertex set $V(G)$ and an edge set $E(X) \subseteq V(X) \times V(X)$, satisfying $(v, v) \notin E(X)$ and $(v, w) \in E(X) \implies (w, v) \in E(X)$. Given two graphs X and Y, a *graph homomorphism from X to Y* is a function $f : V(X) \to V(Y)$ with the property that $(v, w) \in E(X) \implies (f(v), f(w)) \in E(Y)$. We write $X \to Y$ to indicate that there exists a graph homomorphisms from X to Y. Graph homomorphisms encapsulate many familiar graph theoretic parameters. If we let K_c denote the *complete graph* on c vertices, i.e., the graph where every pair of vertices is connected by an edge, then

- the *chromatic number* of X is $\chi(X) = \min\{c : \exists\, X \to K_c\}$,
- the *clique number* of X is $\omega(X) = \max\{c : \exists\, K_c \to X\}$,
- the *independence number* of X is, $\alpha(X) = \max\{c : \exists\, K_c \to \overline{X}\}$,

where $\overline{X}$ denotes the *graph complement* of X, i.e., the graph whose edge set is the complement of X's.

The *graph homomorphism game* from X to Y, which we shall denote by $Hom(X, Y)$, is a synchronous game with inputs $I_A = I_B = V(X)$ and outputs $O_A = O_B = V(Y)$. Alice and Bob win a round provided that whenever they receive inputs that are an edge in X, then their outputs are an edge in Y and that whenever Alice and Bob receive the same vertex in X they produce the same vertex in Y. This is also a synchronous game.

Note that a perfect deterministic strategy for the graph homomorphism game from X to Y is a function $h : V(X) \to V(Y)$ that is a graph homomorphism. In particular, a perfect deterministic strategy exists if and only if $\exists X \to Y$. Similarly, we say that there is a *t-homomorphism* from X to Y and write $X \xrightarrow{t} Y$ if and only if there exists a perfect t-strategy for the graph homomorphism game from X to Y for $t = q$, qs, etc.

Two graphs X and Y are *isomorphic* if and only if there exists a one-to-one onto function $f : V(X) \to V(Y)$ such that (v, w) is an edge in X if and only if $(f(v), f(w))$ is an edge in Y.

We write $X \simeq Y$ to indicate that X and Y are isomorphic. If we let A_X denote the adjacency matrix of X and analogously for A_Y, then it is well-known and easy to check that $X \simeq Y$ if and only if there is a permutation matrix P such that $A_X P = P A_Y$.

The *graph isomorphism game, Iso(X,Y)* between X and Y is a game with the property that two graphs are isomorphic if and only if there exists a perfect deterministic strategy for $Iso(X, Y)$. It was introduced by Atserias et al. [2].

The easiest way to describe the rules for this game is in terms of the *relation* between a pair of vertices. Formally, the relation on a graph is a function $rel : V(X) \times V(X) \to \{0, 1, -1\}$ with

- $rel(v, w) = 0 \iff v = w$,
- $rel(v, w) = -1 \iff (v, w) \in E(X)$,
- $rel(v, w) = +1 \iff v \neq w$ and $(v, w) \notin E(X)$.

We remark that the matrix $S_X := (rel(v, w))_{v,w \in V(X)}$ is known as the *Seidel adjacency matrix* of the graph.

The rules for this game can be stated loosely as requiring that to win, outputs must come from different graphs than inputs, outputs must have the same relations as inputs, and whenever one player's output is the same as the other player's input, then the same must hold for the other player. This final rule says that the deterministic strategy consists of a function and its inverse, instead of just a pair of functions. The input set and output set for this game is the disjoint union of $V(X)$ with $V(Y)$ and

$$\lambda : (V(X) \cup V(Y)) \times (V(X) \cup V(Y)) \to \{0, 1\},$$

satisfies $\lambda(v, w, x, y) = 1$ if and only if the following conditions are met:

- x belongs to a different graph than v and y belongs to a different graph than w,
- if v and w are both vertices of the same graph, then $rel(v, w) = rel(x, y)$.
- if v and w are from different graphs and $x = w$, then $y = v$,

- if v and w are from different graphs and $y = v$, then $x = w$.

Now it is not hard to see that this game is synchronous and it has a perfect deterministic strategy if and only if $X \simeq Y$. Indeed, if it has a perfect deterministic strategy, then there must be a function $f : V(X) \cup V(Y) \to V(X) \cup V(Y)$ and the rules force $v \in V(X) \implies f(v) \in V(Y)$ and $x \in V(Y) \implies f(x) \in V(X)$. Denoting the restrictions of f to $V(X)$ and $V(Y)$ by $f_1 : V(X) \to V(Y)$ and $f_2 : V(Y) \to V(X)$. The fact that $rel(v, w) = rel(f_1(v), f_1(w))$ tells us that f_1 is one-to-one and preserves the edge relationships, since f_2 is also one-to-one, $card(V(X)) = card(V(Y))$ and so both f_1 and f_2 define graph isomorphisms.

We will write $X \simeq_t Y$ if and only if this game has a perfect t-strategy for $t \in \{loc, q, qa, qc, C^*, A^*\}$.

The following result characterizes $\mathcal{A}(Iso(X, Y))$.

Proposition 2.3.1. *Let $X = (V(X), E(X))$ and $Y = (V(Y), E(Y))$ be graphs on n vertices. Then $\mathcal{A}(Iso(X, Y))$ is generated by $4n^2$ self-adjoint idempotents $\{e_{v,w} : v, w \in V(X) \cup V(Y)\}$ satisfying:*

1. $e_{g,g'} = 0$, $\forall g, g' \in V(X)$ *and* $e_{h,h'} = 0$, $\forall h, h' \in V(Y)$,

2. $e_{g,h}^2 = e_{g,h}^* = e_{g,h}$, $\forall g \in V(X), h \in V(Y)$,

3. *for* $g \in V(X)$ *and* $h \in V(Y)$, $e_{g,h} = e_{h,g}$,

4. $\sum_{h \in V(Y)} e_{g,h} = 1$, $\forall g \in V(X)$,

5. $\sum_{g \in V(X)} e_{g,h} = 1$, $\forall h \in V(Y)$,

6. $e_{g,h}e_{g,h'} = 0, \forall h \neq h'$,

7. $e_{g,h}e_{g',h} = 0$, $\forall g \neq g'$,

8. $\sum_{g' : (g,g') \in E(X)} e_{g',h} = \sum_{h' : (h,h') \in E(Y)} e_{g,h'}$, $\forall g, h$.

Proof. The definition of the *-algebra gives us property (2). Similarly, it's quickly clear that (6) and (7) follow from the winning criteria of the game.

To see criteria (1), consider $g, g' \in X$. Then for all $x, y \in X \cup Y$, we have $\lambda(g, x, g', y) = 0$. Therefore, for a fixed x, we have

$$\begin{aligned} e_{g,g'} &= e_{g,g'} \left(\sum_{x \in X \cup Y} e_{y,x} \right) \\ &= \sum_{x \in X \cup Y} e_{g,g'} e_{y,x} \\ &= \sum_{z \in X \cup Y} \lambda(g, x, g', y) e_{g,g'} e_{y,x} \\ &= 0. \end{aligned}$$

So $e_{g,g'} = 0$. Similarly, for $h, h' \in Y$, $e_{h,h'} = 0$.

Criteria (4) and (5) follow easily: for any $g \in X$, then

$$1 = \sum_{x \in X \cup Y} e_{g,x} = \sum_{x \in Y} e_{g,x}.$$

To prove criteria (3), take some $g \in X$ and $h \in Y$ then

$$e_{h,g} = e_{h,g} \left(\sum_{x \in Y} e_{g,x} \right) = \sum_{x \in Y} e_{h,g} e_{g,x} = \sum_{x \in Y} \lambda(h, g, g, x) e_{xg} e_{gx} = e_{hg} e_{gh}.$$

Similarly, $e_{gh} = e_{gh} e_{hg}$. So then

$$e_{gh} = e_{gh}^* = (e_{gh} e_{hg})^* = e_{hg}^* e_{gh}^* = e_{hg} e_{gh} = e_{hg}.$$

Finally to see (8), we have that

$$\sum_{g':(g,g')\in E(X)} e_{g',h} = \Big(\sum_{g':(g,g')\in E(X)} e_{g',h}\Big)\Big(\sum_{h'\in V(Y)} e_{g,h'}\Big) =$$
$$\sum_{g',h':(g,g')\in E(X),h'\in V(Y)} e_{g',h}e_{g,h'} = \sum_{g',h':(g,g')\in E(X),(h,h')\in E(Y)} e_{g',h}e_{g,h'},$$

since $\lambda(g', g, h, h') = 0$ unless $(h, h') \in E(Y)$. Similarly, one shows that $\sum_{h':(h,h')\in E(Y)} e_{g,h'}$ is equal to this latter sum and (8) follows. □

Remark 2.3.2. *A nice compact way to represent the above relations is to consider the $n \times n$ matrix $U = (e_{g,h})_{g\in V(X),h\in V(Y)}$. Then by (2) every entry is a self-adjoint idempotent, while (4) and (5) imply that $U^*U = UU^*$ is the identity matrix, i.e., that U is a unitary. We also, by (6) and (7), have that entries in each row and column are pairwise "orthogonal", i.e., have pairwise 0 product. Such a matrix U will be referred to as a quantum permutation over the $*$-algebra $\mathcal{A}(Iso(X, Y))$.*

Equation (8) implies that $(1 \otimes A_X)U = U(1 \otimes A_Y)$ where A_X and A_Y denote the adjacency matrices of the graphs, and 1 is the unit of the algebra. Thus, Proposition 2.3.1 can be summarized as saying that $\mathcal{A}(Iso(X, Y))$ is the $*$-algebra generated by $\{e_{g,h} : g \in V(X), h \in V(Y)\}$ subject to the relations that $U = (e_{g,h})$ is a quantum permutation with $(1\otimes A_X)U = U(1\otimes A_Y)$. We have that $X \simeq_{A^*} Y$ if and only if a non-trivial $*$-algebra exists satisfying these relations.

Remark 2.3.3. *Combining Proposition 2.3.1 with Theorem 2.2.2, we see that given two graphs X and Y on n vertices:*

- *$X \simeq_q Y$ if and only if there exist a d and projections $E_{g,h} \in M_d$ such that $U = (E_{g,h})$ is a unitary in $M_n(M_d)$ and $(1 \otimes A_X)U = U(1 \otimes A_Y)$,*
- *$X \simeq_{qa} Y$ if and only if there exist projections $E_{g,h} \in \mathcal{R}^\omega$ such that $U = (E_{g,h}) \in M_n(\mathcal{R}^\omega)$ is a unitary and $(1 \otimes A_X)U = U(1 \otimes A_Y)$,*
- *$X \sim_{qc} Y$ if and only if there exists projections $E_{g,h}$ in some C$*$-algebra $\mathcal{A}$ with a trace such that $U = (E_{g,h}) \in M_n(\mathcal{A})$ is a unitary and $(1 \otimes A_X)U = U(1 \otimes A_Y)$,*

- *$X \simeq_{C^*} Y$ if and only if there exists projections $E_{g,h}$ on a Hilbert space H such that $U = (E_{g,h}) \in M_n(B(H))$ is a unitary and $(1 \otimes A_X)U = U(1 \otimes A_Y)$.*

Also, if there exists a unital $*$-homomorphism from $\pi : \mathcal{A}(Iso(X, Y)) \to \mathbb{C}$, then $(\pi(e_{g,h})) \in M_n$ will be a permutation matrix, satisfying $A_X(\pi(e_{g,h})) = (\pi(e_{g,h}))A_Y$, which is the classical notion of isomorphism for graphs.

Note that we have the following obvious implications.

$$X \cong Y \implies X \cong_q Y \implies X \cong_{qa} Y \implies X \cong_{qc} Y \implies X \cong_{C^*} Y \implies X \cong_{A^*} Y.$$

Moreover, it is known that the first two implications are not reversible [2, 26]. The question of whether the third implication holds is still open. The question whether the implications $X \cong_{A^*} Y \implies X \cong_{C^*} Y \implies X \cong_{qc} Y$ hold for generic X and Y had remained open for quite some time. Only very recently the implication $C^* \implies qc$ was obtained in [30]. One of our main results is that the implication $A^* \implies qc$ holds. In other words, $\mathcal{A}(Iso(X, Y)) \neq 0$ if and only if $\mathcal{A}(Iso(X, Y))$ admits a tracial state. This is somehow surprising, because the same conclusion cannot be made for the algebras $\mathcal{A}(Hom(X, Y))$ [24].

2.4 The Linear Binary Constraint System Game

The Linear Binary Constraint Synchronous Game (LBCS) game was first introduced in [26] and is a synchronous version of what is classically known as the Linear Binary Constraints System (LBC) game. Given an $m \times n$ matrix $A = (a_{i,j})$ over the field of two elements, $\mathbb{Z}_2$ and a vector b, we introduce a game denoted $syncBCS(A, b)$, that is intended to convince a referee that Alice and Bob have a solution x to the equation $Ax = b$.

For $i = 1, \ldots, m$, let $V_i = \{j : a_{i,j} \neq 0\}$. Note that to solve the i-th equation in $Ax = b$, we only need

$$\sum_{j \in V_i} a_{i,j} x_j = b_i,$$

since the remaining terms are irrelevant. Set

$$S_i^b = \{x \in \mathbb{Z}_2^n : \sum_{j \in V_i} a_{i,j} x_j = b_i \text{ and } x_j = 0 \text{ for } j \notin V_i\}.$$

We associate a synchronous game to $Ax = b$ as follows:

Definition 2.4.1. *Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}_2$ and $b \in \mathbb{Z}_2^n$. The synchronous BCS game associated to $Ax = b$, denoted $synBCS(A, b)$, is given as follows:*

1. *the input set is $\mathcal{I} = \{1, \ldots, m\}$;*

2. *the output set is $\mathcal{O} = \mathbb{Z}_2^n$;*

3. *given input (i, j), Alice and Bob win on output (x, y) if and only if $x \in S_i^b$, $y \in S_j^b$, and for all $k \in V_i \cap V_j$, $x_k = y_k$.*

The connections between the graph isomorphism game and the synchronous binary constraint system game are further explored in Section 4.2.3.

2.5 The Metric Isometry Game

A *finite metric space* is a finite set X equipped with a finite metric $d : X \times X \to [0, \infty)$. Throughout this paper, we assume all metric spaces are finite, unless stated otherwise. Given two finite metric spaces (X, d_X) and (Y, d_Y), we say the two metric spaces are *isometric* if there is a bijection f between X and Y that preserves distances, that is, $d_X(x, x') = d_Y(f(x), f(x'))$ for all $x, x' \in X$. If they are isometric, we write $X \simeq Y$.

The metric isometry game, $Isom(X, Y)$ is a modification of the graph isomorphism game. To define $Isom(X, Y)$, we set the inputs/outputs to be $I = O = X \sqcup Y$. Suppose the inputs for Alice and Bob are v and w while the outputs are a and b, respectively. The players win the round if all of the following are satisfied:

1. v and a are from different spaces

2. w and b are from different spaces

3. If v and w are from the same metric space, then $d_\bullet(v, w) = d_\bullet(a, b)$

4. If v and w are from different spaces, then $v = b$ if and only if $w = a$.

Note that condition 3 along with the fact that $d(v, w) = 0$ iff $v = w$ forces the game $Isom(X, Y)$ to be bi-synchronous. We may recast $Isom(X, Y)$ in terms of graphs, via a modification of the graph isomorphism game for weighted graphs.

From a metric space (X, d), one can derive a weighted graph $G_X = (V(G), E(G), w)$ arising from the metric space. We let $V(G) = X$ and let $E(G) = X \times X$ so that G is the complete graph on $|X|$ vertices. We set the weight of the edge between x and y to be $d(x, y) = d(y, x)$.

We note immediately that $d(x, x) = 0$ implies that the graph has no loops, $d(x, y) = d(y, x)$ for all $x, y \in X$ implies that the graph is undirected, and the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ implies that the "cheapest" way to get from x to z (or vice versa) is directly.

The distance matrix for this graph, $A_X = [a_{ij}^X]_{i,j \in X}$, is given by $a_{ij} = d(i, j)$. It is a symmetric matrix with zeros along the diagonal.

We will now define the *weighted graph isomorphism game*, an expansion of the well-studied graph isomorphism game and show it is analogous to the Metric Isometry Game described earlier. We start the game with two simple, weighted graphs, G and H.

Definition 2.5.1. *From a simple weighted graph G, we define the minimum complete graph G' to be the complete graph on the same vertices with weight between vertices in G' to be the cheapest path weight between the two vertices in G.*

Remark 2.5.2. *For any simple weighted graph G, the minimum complete graph G' gives rise to a metric space in a manner analogous to the process earlier. It is possible that two distict simple weighted graphs produce the same minimum complete graph.*

From graphs G and H, we obtain the minimum complete graphs G' and H'. We set the inputs and outputs for the weighted graph isomorphism game to be $I = O = V(G) \sqcup V(H) = V(G') \sqcup V(H')$.

The referee will give two inputs, v and w to the two players, respectively Alice and Bob. They will reply with the outputs a and b. We say that the players win that round if the following criteria are satisfied:

1. v and a are from different graphs

2. w and b are from different graphs

3. If v and w are from the same graph, then $d.(v,w) = d.(a,b)$ where the distance is the minimum path length of the minimum complete graphs

4. If v and w are from different graphs, then $v = b$ if and only if $w = a$.

The directed graph isomorphism game is a reformulation of the metric isometry game.

Theorem 2.5.3. *Take two metric spaces (X, d_X) and (Y, d_Y) and let G' and H' be the corresponding minimum complete graphs.*

1. *(X, d_X) and (Y, d_Y) are isometric*

2. *The Metric Isometry Game played on (X, d_X) and (Y, d_Y) has a winning classical strategy*

3. *The minimum complete graphs G' and H' are isomorphic*

4. *The Weighted Graph Isomorphism game for G' and H' has a winning classical strategy*

Proof. $(1) \Leftrightarrow (3)$ is straightforward to check.

$(1) \Rightarrow (2)$: Suppose (X, d_X) and (Y, d_Y) are isometric, and $\varphi : X \to Y$ is an isomorphism. If the player receives point $x \in X$, then they should respond with $\varphi(x)$ and similarly, if the player receives vertex $y \in Y$, then they should respond with $\varphi^{-1}(y)$. This will win the $(X, d_X) - (Y, d_Y)$ metric isometry game and it is indeed a classical strategy.

$(2) \Rightarrow (1)$: Since the metric isometry game is a synchronous game, there must exist a winning deterministic strategy. Indeed, let Alice's answers be given by the function $f_A : X \sqcup Y \to X \sqcup Y$, and let Bob's answers be given by the function $f_B : X \sqcup Y \to X \sqcup Y$. Since the game is synchronous, the two functions must be equal and so call this function $f := f_A = f_B$.

Note that the restriction $f|_X : X \to Y$ is an isomorphism from X to a subset of Y. Similarly, the restriction $f|_Y : Y \to X$ is an isomorphism from Y to a subset of X. This tells us that X and Y are isometric and that $f|_X$ and $f|_Y$ are isomorphisms.

We are left to show that $f|_X = f|_Y^{-1}$. That is, we want to show that for all $x \in X$, $x = f|_Y f|_X(x)$. Consider the case where Alice receives $x \in X$ and Bob receives $f(x)$. The deterministic strategy dictates that Alice will respond with $f(x)$, and so because $y_A = x_B$ then the winning strategy criteria implies that $x_A = y_B$ and so Bob is forced to respond with x. This is true for all $x \in X$, and so $f|_X = f|_Y^{-1}$.

The proof of $(3) \Leftrightarrow (4)$ is the same as the proof above. $\square$

We next want to know what the $*$-algebra of the metric isometry game is. This theorem mimics Proposition 2.3.1.

Theorem 2.5.4. *Let (X, d_X) and (Y, d_Y) be metric spaces, each with size n. Then $\mathcal{A}(Isom(X, Y))$ is generated by $4n^2$ self-adjoint idempotents $\{e_{z,w} \mid z, w \in X \sqcup Y\}$ satisfying*

1. $e_{x,x'} = 0$ *for all* $x, x' \in X$ *and* $e_{y,y'} = 0$ *for all* $y, y' \in Y$
2. $e_{x,y}^2 = e_{x,y}^* = e_{x,y}$ *for all* $x \in X, y \in Y$
3. *for* $x \in X$ *and* $y \in Y$, $e_{x,y} = e_{y,x}$
4. $\sum_{y \in Y} e_{x,y} = 1$ *for all* $x \in X$
5. $\sum_{x \in X} e_{x,y} = 1$ *for all* $y \in Y$
6. $e_{x,y}e_{x,y'} = 0$ *for all* $y \neq y'$
7. $e_{x,y}e_{x',y} = 0$ *for all* $x \neq x'$

8. for any $x \in X$ and $y \in Y$, then

$$\sum_{x' \in X} d_X(x,x')e_{x',y} = \sum_{y' \in Y} d_Y(y',y)e_{x,y'}$$

Proof. The proof from Proposition 2.3.1 proves (1) through (7). To see (8), for all $x \in X$ and $y \in Y$, and recalling that $\lambda(x',x,y,y') = 1$ if and only if $d_X(x,x') = d_Y(y,y')$, we see that

$$\begin{aligned}
\sum_{x' \in X} d_X(x,x')e_{x',y} &= \sum_{x' \in X} d_X(x,x')e_{x',y} \left(\sum_{y' \in Y} e_{x,y'} \right) \\
&= \sum_{x' \in X, y' \in Y} d_X(x,x')e_{x',y}e_{x,y'} \\
&= \sum_{x' \in X, y' \in Y} d_X(x,x')\lambda(x',x,y,y')e_{x',y}e_{x,y'} \\
&= \sum_{x' \in X, y' \in Y} d_Y(y',y)e_{x',y}e_{x,y'} \\
&= \left(\sum_{x' \in X} e_{x',y} \right) \sum_{y' \in Y} d_Y(y',y)e_{x,y'} \\
&= \sum_{y' \in Y} d_Y(y',y)e_{x,y'}
\end{aligned}$$

□

Remark 2.5.5. *Let $U = [e_{x,y}]_{x \in X, y \in Y}$. Then the above relations imply that U is a magic unitary matrix and that $(1 \otimes D_X)U = U(1 \otimes D_Y)$. Equivalently, for the weighted graph isomorphism for minimal complete corresponding graphs G', H' we have that $(1 \otimes A_{G'})U = U(1 \otimes A_{H'})$ where we note that the adjacency matrices for the graphs G' and H' are identical to the distance matrix for their corresponding metrics.*

Thus, the game $*$-algebra $\mathcal{A}(Isom(X,Y))$ can be viewed a non-commutative analogue of the space of isometries from X to Y.

Definition 2.5.6. *Motivated by Theorem 2.2.2, for two metric spaces (X, d_X) and (Y, d_Y) we define*

- *$X \cong_q Y$ if and only if there exists d and projections $E_{x,y} \in M_d$ such that $U = (E_{x,y})$ is a unitary in $M_n(M_d)$ and $(1 \otimes D_X)U = U(1 \otimes D_Y)$.*

- *$X \cong_{qa} Y$ if and only if there exists projections $E_{x,y} \in \mathcal{R}^\omega$ such that $U = (E_{x,y}) \in M_n(\mathcal{R}^\omega)$ is a unitary and $(1 \otimes D_X)U = U(1 \otimes D_Y)$. Here, R^ω is the hyperfinite II_1 factor, and interested readers can learn more in [1].*

- *$X \cong_{qc} Y$ if and only if there exists projections $E_{x,y}$ in some C^*-algebra $\mathcal{A}$ with a tracial state such that $U = (E_{x,y}) \in M_n(\mathcal{A})$ is a unitary and $(1 \otimes D_X)U = U(1 \otimes D_Y)$.*

- *$X \cong_{C*} Y$ if and only if there exists projections $E_{x,y}$ in some Hilbert space $\mathcal{H}$ such that $U = (E_{x,y}) \in M_n(B(\mathcal{H}))$ is a unitary and $(1 \otimes D_X)U = U(1 \otimes D_Y)$.*

Remark 2.5.7. *Given two metric spaces (X, d_X) and (Y, d_Y) with corresponding minimal complete graphs G', H', then since the two game $*$-algebras $\mathcal{A}(Isom(X, Y))$ and $\mathcal{A}(Iso(G', H'))$ are the same, we can see that, using the notation from [10], for any $t \in \{loc, q, qa, qc, C^*, A^*\}$ we have that $X \cong_t Y$ if and only if $G' \cong_t H'$.*

3. QUANTUM SETS, GRAPHS, AND THEIR QUANTUM AUTOMORPHISM GROUPS

The examples of CQGs that feature in this paper are the quantum automorphism groups of certain finite structures, such as sets, graphs, and their quantizations. In order to describe these objects, we first quantize the notion of a (measured) finite set, then proceed to quantum graphs. All of the definitions that follow are quite standard in the operator algebra literature [42, 3, 4, 38]. The idea of a quantum set or a quantum graph also appears in [31, 32] using the language of special symmetric dagger Frobenius algebras.

3.1 Quantum sets and graphs

Definition 3.1.1. *A (finite, measured) quantum set is a pair $X = (\mathcal{O}(X), \psi_X)$, where $\mathcal{O}(X)$ is a finite dimensional C*-algebra and $\psi_X : \mathcal{O}(X) \to \mathbb{C}$ is a faithful state.*

We write $|X|$ for $\dim \mathcal{O}(X)$, and refer to this value as the cardinality or size of X.

The reason for our choice of notation is that when $\mathcal{O}(X)$ is commutative, Gelfand theory tells us that we are really just talking about a finite set X (the spectrum of $\mathcal{O}(X)$) equipped with a probability measure μ_X defined $\psi_X(f) = \int_X f(x) d\mu_X(x)$ for each $f \in \mathcal{O}(X)$.

Let $X = (\mathcal{O}(X), \psi_X)$ be a quantum set. Let $m_X : \mathcal{O}(X) \otimes \mathcal{O}(X) \to \mathcal{O}(X)$ and $\eta_X : \mathbb{C} \to \mathcal{O}(X)$ be the multiplication and unit maps, respectively. In what follows, we will generally only be interested in a special class of finite quantum sets – namely those that are measured by a *δ-form* ψ_X, which we now define:

Definition 3.1.2 ([4]). *Let $\delta > 0$. A state $\psi_X : \mathcal{O}(X) \to \mathbb{C}$ is called a δ-form [4] if*

$$m_X m_X^* = \delta^2 \iota,$$

where the adjoint is taken with respect to the Hilbert space structure on $\mathcal{O}(X)$ coming from the GNS construction with respect to ψ_X.

For purposes of distinguishing between the Hilbert and C*-structures on $\mathcal{O}(X)$, we denote this

Hilbert space by $L^2(X)$.

The most basic examples of δ-forms are given by the uniform measure on the n-point set $X = [n]$ and the canonical normalized trace on $M_n(\mathbb{C})$. In the first case, a simple calculation shows that $m^*(e_i) = ne_i \otimes e_i$, where $(e_i = e_i^* = e_i^2)_{i=1}^n$ is the standard basis of projections for $\mathcal{O}(X)$, and so we have $\delta = \sqrt{n}$. In the second case, one can show that $m^*(e_{ij}) = n\sum_{k=1}^n e_{ik} \otimes e_{kj}$, where $(e_{ij})_{1\leq i,j\leq n}$ are the matrix units for $M_n(\mathbb{C})$. So in this case we have $\delta = n$. More generally, if we have a multimatrix decomposition $\mathcal{O}(X) = \bigoplus_{i=1}^s M_{n(i)}(\mathbb{C})$ and $\psi_X = \bigoplus_{i=1}^s \mathrm{Tr}(Q_i \cdot)$ is a faithful state (so $0 < Q_i \in M_{n(i)}(\mathbb{C})$ and $\sum_i \mathrm{Tr}(Q_i) = 1$), then ψ_X is a δ-form if and only if $\mathrm{Tr}(Q_i^{-1}) = \delta^2$ for all $1 \leq i \leq s$. In particular, $\mathcal{O}(X)$ admits a unique tracial δ-form with $\delta^2 = \dim \mathcal{O}(X)$ given by $\psi_X = \bigoplus_{i=1}^s \frac{n(i)}{|X|}\mathrm{Tr}(\cdot)$.

Convention 3.1.3. *Unless otherwise stated, we assume from now on that the quantum sets we consider are equipped with δ-forms.*

We now endow quantum sets with an additional structure of a quantum adjacency matrix, turning them into quantum graphs. The following definition of a quantum adjacency matrix/graph is a generalization of the [31, Definition 5.1] to our framework.

Definition 3.1.4. *Let X be a quantum set equipped with a δ-form ψ_X. A self-adjoint linear map $A_X : L^2(X) \to L^2(X)$ is called a quantum adjacency matrix if it has the following properties*

1. $m_X(A_X \otimes A_X)m_X^* = \delta^2 A_X$.

2. $(\iota \otimes \eta_X^* m_X)(\iota \otimes A_X \otimes \iota)(m_X^* \eta_X \otimes \iota) = A_X$

3. $m_X(A_X \otimes \iota)m_X^* = \delta^2 \iota$

We call the triple $X = (\mathcal{O}(X), \psi_X, A_X)$ a quantum graph.

Remark 3.1.5. *In the special case where $\mathcal{O}(X)$ is equipped its unique tracial δ-form, then the definition of a quantum graph given here is equivalent to the one given in [31]. In addition, as explained in [31], a quantum graph $X = (\mathcal{O}(X), \psi_X, A_X)$, where $\mathcal{O}(X)$ is a commutative C*-algebra, captures precisely the notion of a classical graph. Indeed, in this case the spectrum X*

of $\mathcal{O}(X)$ is a finite set and ψ_X is the uniform probability measure on X. If we write A_X as a matrix $A_X = [a_{ij}]_{i,j\in X}$ with respect to the canonical orthonormal basis of normalized projections $(\sqrt{n}e_i)_{i=1}^n \subset L^2(X)$, then conditions (1), (2) *and* (3) *say, respectively, that*

$$a_{ij}^2 = a_{ij}, \quad a_{ij} = a_{ji}, \quad a_{ii} = 1 \qquad (i, j \in X).$$

In other words, X is the vertex set of a classical graph (as defined in Section 2.3) with adjacency matrix $A_X - I_n$. Thus, in the quantum definition of a graph, we choose to work with reflexive graphs ($(v, v) \in E(X)\ \forall v \in V(X)$). This choice is purely cosmetic from the perspective of (quantum) symmetries of graphs, in the sense that we have a bijection between (quantum) symmetries of reflexive graphs and those of their irreflexive versions.

Remark 3.1.6. *Note that any quantum set X equipped with a δ-form ψ_X can be trivially upgraded to a quantum graph in two ways. The first way is by declaring $A_X = \delta^2\psi_X(\cdot)1$. The second is by declaring $A_X = \iota$. In the case of classical finite sets X, these constructions correspond to the complete graph $K_{|X|}$ and its (reflexive) complement $\overline{K_{|X|}}$, respectively. For general quantum sets X equipped with the quantum adjacency matrix $A_X = \delta^2\psi_X(\cdot)1$, we will call these graphs quantum complete graphs. For a general quantum graph X, we can also talk about its (reflexive) complement $\overline{X}$, which is given by $\overline{X} = (\mathcal{O}(X), \psi_X, A_{\overline{X}})$ with $A_{\overline{X}} = \delta^2\psi_X(\cdot)1 + \iota - A_X$. With this definition we have that the complement of a quantum complete graph X is the "edgeless" quantum graph $\overline{X} = (\mathcal{O}(X), \psi_X, \iota)$.*

Remark 3.1.7. *An equivalent definition of a quantum graph has been defined in [43, 44]: a quantum graph is a triple $(\mathcal{S}, \mathcal{M}, M_n)$ where $\mathcal{M}$ is a non-degenerate von Neumann algebra and $\mathcal{M} \subseteq M_n$, $\mathcal{S} \subseteq M_n(\mathbb{C})$ is an operator system and $\mathcal{S}$ is an $\mathcal{M}' - \mathcal{M}'$-bimodule with respect to matrix multiplication.*

To see the connection between these two definitions, we fix the tracial δ-form ψ and we can set $\mathcal{M} = \mathcal{B} \subseteq B(L^2(X))$. We then set the $\mathcal{M}' - \mathcal{M}'$ bimodule $\mathcal{S}$ to be $P(B(L^2(X)))$ where P is the projection mapping the operator $T \in B(L^2(X))$ to $\delta^{-2}m_X(A_X \otimes T)m_X^*$. However, the relation *between the two definitions is not one-to-one, as two distinct quantum graphs $(\mathcal{B}, \psi_X, A_X)$ in the sense of [31] can yield the same $\mathcal{M}' - \mathcal{M}'$-bimodule $\mathcal{S}$.*

We now introduce the quantum automorphism groups of quantum graphs. The definition of these quantum automorphism groups follows along the same lines as for the quantum automorphism groups of classical graphs [5] and also the quantum automorphism groups of quantum sets [42, 3, 4].

Definition 3.1.8. *Let $X = (\mathcal{O}(X), \psi_X, A_X)$ be a quantum graph with $n = |X|$ and fix an orthonormal basis $\{e_i\}_{i=1}^n$ for $L^2(X)$. Define $\mathcal{O}(G_X)$ to be the universal unital $*$-algebra generated by the coefficients u_{ij} of a unitary matrix $u = [u_{ij}]_{i,j=1}^n \in M_n(\mathcal{O}(G_X))$ subject to the relations making the map*

$$\rho_X : \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G_X); \qquad \rho_X(e_i) = \sum_k e_j \otimes u_{ji}$$

a unital $*$-homomorphism satisfying the A_X-covariance condition $\rho_X(A_X \cdot) = (A_X \otimes \iota)\rho_X$.

The notation $\mathcal{O}(G_X)$ is meant to convey the notion that the algebra consists of representative functions on a CQG G_X. Specifically, it is the "largest" CQG acting on X so as to preserve the measure ψ_X and graph structure A_X. This is formalized in the following result, whose proof is a straightforward application of the universality implicit in Definition 3.1.8.

Proposition 3.1.9. *The $*$-algebra $A = \mathcal{O}(G_X)$ admits a Hopf $*$-algebra structure defined by*

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad S(u_{ij}) = u_{ji}^*, \quad \epsilon(u_{ij}) = \delta_{ij} \qquad (1 \le i, j \le n).$$

Furthermore, the action of G_X on X given by ρ_X preserves ψ_X in the sense that

$$(\psi_X \otimes \iota)\rho_X = \psi_X(\cdot)1 : \mathcal{O}(X) \to \mathcal{O}(G_X).$$

We call G_X the quantum automorphism group of the quantum graph X.

Proof. This is a direct computation that we leave to the reader. In fact a proof of this result will also follow as special case of the more general arguments presented following Remark 4.2.3.

□

Remark 3.1.10. *Quantum automorphism groups are natural quantum analogues of their classical counterparts. Indeed, the abelianization of $\mathcal{O}(G_X)$ is exactly $\mathcal{O}(Aut(X))$, the algebra of complex-valued functions on the group of automorphisms of the classical graph X.*

Example 3.1.11. *When X is a quantum complete graph, then G_X is none other than Wang's quantum automorphism group of the finite space $(\mathcal{O}(X), \psi_X)$ [42, 4]. In particular, the quantum automorphism group of the classical complete graph K_n is precisely the quantum symmetric group S_n^+ of Example 1.3.3.*

3.2 W^*-quantum metric spaces

The definitions of a W^*-quantum metric space and the theorems that follow in this section were introduced in [27]. They have since been studied in [14].

A non-commutative analogue of a metric space, $\mathcal{V} = \{\mathcal{V}_t\}_{t\geq 0}$, was defined in [27] using the language of von Neumann algebras, called a W^*-quantum metric. The intuition behind their definition is that each family $\mathcal{V}_t$ is a non-commutative analogue of pairs of points (x, y) whose distance is at most t, while motivation for this definition comes primarily from the standard model of quantum error correction. The definition of a W^*-quantum metric is related to other models of quantum metric spaces: Connes' notion of a spectral triple produces a W^*-quantum metric [18], and every W^*-metric produces Reiffel's Lipschitz seminorm [36].

Definition 3.2.1. *([27], Definition 2.3) A W^*- quantum metric on a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is a one-parameter family of weak* closed operator systems $\mathcal{V}_t \subseteq B(\mathcal{H})$, $t \in [0, \infty)$ such that*

1. $\mathcal{V}_s\mathcal{V}_t \subseteq \mathcal{V}_{s+t}$ *for all* $s, t \geq 0$

2. $\mathcal{V}_t = \cap_{s>t}\mathcal{V}_s$ *for all* $t \geq 0$

3. $\mathcal{V}_0 = \mathcal{M}'$ where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ inside $B(\mathcal{H})$

We say a W^*-quantum metric space is the pair $(\mathcal{M}, \mathcal{H}, \{\mathcal{V}_t\}_{t\geq 0})$ of a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ together with a W^*-quantum metric $\{\mathcal{V}_t\}$.

It is easy to see that the $\mathcal{V}_t$ are nested. It can also be seen that $\mathcal{V}_0$ is a von Neumann algebra.

Given a (possibly infinite) metric space (X, d), we can view the classical metric space as an example of a W^*-quantum metric on an abelian von Neumann algebra. We take the von Neumann algebra $\mathcal{M} = \ell^\infty(X)$ of bounded multiplication operators on $\ell^2(X)$ and define $\{\mathcal{V}_t^X\}$ by

$$\begin{aligned}\mathcal{V}_t^X &= \overline{\text{span}}^{wk*}\{V_{xy} \in \mathcal{B}(\ell^2(X)) \mid d(x,y) \leq t\} \\ &= \{A \in B(\ell^2(X)) \mid \langle Ae_y, e_x\rangle = 0 \text{ if } d(x,y) > t\}\end{aligned}$$

where $V_{x,y} \in B(\ell^2(X))$ is the rank one operator $V_{xy} : g \mapsto \langle g, e_y\rangle e_x$ and $\{e_x\}_{x\in X}$ is the standard orthonormal basis on $\ell^2(X)$.

Proposition 3.2.2. *([27], Proposition 2.5.) The construction above gives us a W^*-quantum metric space.*

Conversely, if we have a W^*-quantum metric $\{\mathcal{V}_t\}$ on the commutative von Neumann algebra $\mathcal{M} = \ell^\infty(X)$, then we may set

$$d(x,y) = \inf\{t \mid \langle Ae_y, e_x\rangle \neq 0 \text{ for some } A \in \mathcal{V}_t\}$$

to obtain a metric on X. Thus, we have obtained a correspondence between W^*-quantum metrics on abelian von Neumann algebras and classical metric spaces.

To motivate Definition 3.2.1, given a classical metric space (X, d) we may look at the family of relations given by $R_t = \{(x,y) \in X \times X \mid d(x,y) \leq t\}$. There is then the following correspondence between a classical metric space, this family of relations, and any quantum metric space as defined above:

$$
\begin{array}{lcccc}
d(x,x) = 0 & \leftrightarrow & R_0 \text{ is the diagonal relation} & \leftrightarrow & I \in \mathcal{M}' = \mathcal{V}_0 \\
d(x,y) = d(y,x) & \leftrightarrow & R_t = R_t^T & \leftrightarrow & \mathcal{V}_t = \mathcal{V}_t^* \\
d(x,z) \leq d(x,y) + d(y,z) & \leftrightarrow & R_s R_t \subseteq R_{s+t} & \leftrightarrow & \mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t}
\end{array}
$$

where R_t^T denotes the transpose of the relation, that is, $(x,y) \in R_t$ if and only if $(y,x) \in R_t^T$. We can also note here that the relations R_t are nested, as are the $\mathcal{V}_t$. From a family of relations $\{R_t\}_{t\geq 0}$ with the properties above, we can see that the relations define a unique metric $d(x,y) = \inf\{t \mid (x,y) \in R_t\}$.

Example 3.2.3. *One can obtain a W^*-quantum metric from a classical graph $G = (V(G), E(G))$. If $|V(G)| = n$ then we may equip the space $V(G)$ with the shortest path metric coming from the graph. We then set*

$$\mathcal{V}_1 = \operatorname{span}\{E_{ij} \mid i = j \text{ or } i \text{ is adjacent to } j\} \subseteq M_n(\mathbb{C}) = B(\ell^2(V(G)))$$

where $E_{ij} \in M_n(\mathbb{C})$ is the matrix of all zeros with a one in the (i,j) entry. We can then set the larger sets to be

$$\mathcal{V}_k = \mathcal{V}_1^k = \operatorname{span}\{A_1 \ldots A_k \mid A_1, \ldots, A_k \in \mathcal{V}_1\} \subseteq M_n(\mathbb{C}).$$

For any non-integer t, we set $\mathcal{V}_t = \mathcal{V}_{\lfloor t \rfloor}$. It is not hard to check that this gives us a W^*-quantum metric on the von Neumann algebra $\mathcal{M} = M_n(\mathbb{C}) = B(\ell^2(V(G)))$.

A similar argument holds for the class of quantum graphs, an operator space generalization of classical graphs.

Example 3.2.4. *Our goal is to obtain a W^*-quantum metric from a quantum graph $X = (\mathcal{S}, \mathcal{M}, M_n)$. We set $\mathcal{V}_0 = \mathcal{M}'$, and if we assume $\mathcal{S}$ is non-reflexive, then $\mathcal{V}_1 = \mathcal{S}$ is orthogonal to $\mathcal{M}'$, that is, $\mathcal{V}_1 \subseteq \mathcal{M}'^{\perp}$.*

Once we have $\mathcal{V}_1$, then we may set $\mathcal{V}_k = \mathcal{V}_1^k$ as before.

This connects to the quantum adjacency matrix definition: for operators T, we may consider the compression $\frac{1}{\delta^2}m(A_X \otimes T)m^*$ which classically corresponds to the Schur multiplication $A_X \cdot T$. When we think of Schur multiplication by the adjacency matrix, it produces $\mathcal{V}_1$, that is, it kills all matrix units that aren't adjacent.

3.3 Quantum isometry group of W^*-quantum metric spaces

The aim of this section is to generalize Banica's construction of the quantum isometry group for classical metric spaces to the class of W^*-quantum metric spaces. We define the quantum isometry group of the W^*-quantum metric spaces, answering the question which has been asked in [23].

We first recall Banica's quantum isometry group for finite metric spaces.

Definition 3.3.1. *([5]) The quantum isometry group of a finite metric space (X, d) is defined to be $\mathcal{A} = \mathcal{O}(G^X)$ where $G^X = (\mathcal{A}, \Delta)$ is the quotient of $\mathcal{O}(S_n^+)$ by the ideal generated by the relations $UD = DU$, where $D = [d(x, y)]_{x,y \in X}$ is the distance matrix. That is,*

$$\mathcal{O}(G^X) = \mathcal{O}(S_n^+)/\langle UD = DU \rangle.$$

Comultiplication is given by $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which maps $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$.

3.3.1 Actions of a quantum group on a W^*-quantum metric

Every compact quantum group is equipped with a unique left invariant weight, called the Haar weight and denoted by h. For a compact quantum group G, we denote $C_r(G)$ to be the corresponding reduced C^*-algebra, that is, the image of $C(G)$ under the GNS representation $\pi_h : C(G) \to B(L^2(G))$. We equip $C_r(G)$ with a comultiplication Δ. We denote by $L^\infty(G)$ to be the von Neumann algebra generated by $C_r(G)$ in $B(L^2(G))$ and the extension of the corresponding comultiplication $L^\infty \to L^\infty(G)\overline{\otimes}L^\infty(G)$ will be denoted Δ_G. An interested reader can see [29, 28, 33] for more information.

Definition 3.3.2. *Given a compact quantum group G and a von Neumann algebra $\mathcal{M}$, an action of G on $\mathcal{M}$ is a normal injective unital $*$-homomorphism $\alpha : \mathcal{M} \to \mathcal{M}\overline{\otimes}L^\infty(G)$ that satisfies the action equation*

$$(\alpha \otimes \mathrm{id}_{\mathcal{M}}) \circ \alpha = (\mathrm{id}_{L^\infty(G)} \otimes \Delta_G) \circ \alpha.$$

Definition 3.3.3. *Let the compact quantum group G act on a von Neumann algebra $\mathcal{M}$ by $\alpha : \mathcal{M} \to \mathcal{M}\overline{\otimes}L^\infty(G)$. We call a state ψ α-invariant if $(\psi \otimes \mathrm{id})\alpha = \psi(\cdot)1$.*

It's known that we are guaranteed to have an invariant state in certain circumstances [41].

Proposition 3.3.4. *Consider a von Neumann algebra $\mathcal{M}$ with a faithful state ϕ and a compact quantum group G. If the compact quantum group G acts on $\mathcal{M}$ by $\alpha : \mathcal{M} \to \mathcal{M}\overline{\otimes}L^\infty(G)$, then there exists a (not necessarily unique) α-invariant state ψ with $(\psi \otimes 1)\alpha(x) = \psi(x)1$.*

The proof follows by letting $\psi(x) = (\phi \otimes h)\alpha(x)$ where h is the Haar measure and one can view this as an average relative to the action $G \curvearrowright^\alpha \mathcal{M}$. Since ϕ is faithful, one can show ψ is also faithful.

To motivate the next definition, we note that given a representation $U \in M(K(\mathcal{H}) \otimes C(G))$, we automatically get an action

$$\begin{aligned} \alpha : B(\mathcal{H}) &\to B(\mathcal{H})\overline{\otimes}L^\infty(G) \\ T &\mapsto U^*(T \otimes 1)U. \end{aligned} \tag{3.1}$$

Definition 3.3.5. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let G be a compact quantum group. Given a von Neumann algebraic action*

$$\alpha : \mathcal{M} \to \mathcal{M}\overline{\otimes}L^\infty(G)$$

we say the action is unitarily implemented if there exits a unitary representation $U \in M(K(\mathcal{H}) \otimes C_r(G))$ such that

$$\alpha(x) = U^*(x \otimes 1)U \qquad x \in \mathcal{M}.$$

It was shown in [41] that there always exists a unitarily implemented action from $\alpha : \mathcal{M} \to \mathcal{M}\overline{\otimes}L^\infty(G)$ for free if we have an invariant state. By Prop 3.3.4, if we have the action α and fix any faithful state then we can obtain a faithful α-invariant state. With respect to the GNS Hilbert space for $\mathcal{M}$, we can realize this as unitarily implemented by averaging the state with the Haar measure.

We may then naturally extend such a unitarily implemented action given by Equation (3.1) to an action on $B(\mathcal{H})$.

Definition 3.3.6. *Given a W^*-quantum metric space $(\mathcal{M}, \mathcal{H}, \mathcal{V}_t)$ with $\mathcal{M} \subseteq B(\mathcal{H})$ and a compact quantum group G, we say that G acts on the W^*-quantum metric space if there is a unitarily implemented, von Neumann algebraic action $\alpha : \mathcal{M} \to \mathcal{M} \otimes L^\infty(G)$ such that*

$$\alpha(\mathcal{V}_t) \subseteq \mathcal{V}_t\overline{\otimes}L^\infty(G) \qquad \forall t \geq 0$$

Remark 3.3.7. *For every compact quantum group, there exists the universal C^*-algebra associated to G, denoted $C^u(G)$. Whenever one has some representation $U \in M(\mathcal{K}(\mathcal{H}) \otimes C_r(G))$, there exists a corresponding representation of the universal version $\widehat{U} \in M(\mathcal{K}(\mathcal{H}) \otimes C^u(G))$. See Proposition 3.13. in [28] for more details.*

Definition 3.3.8. *Given a possibly infinite dimensional W^*-quantum metric space $(\mathcal{M}, \mathcal{H}, \mathcal{V}_t)$, we define a universal compact quantum group $G^{\mathcal{V}}$ acting on the quantum metric space by requiring the following two properties:*

1. *$C^u(G^{\mathcal{V}})$ is generated by a fundamental representation $\mathbb{U} \in M(K(\mathcal{H}) \otimes C^u(G^{\mathcal{V}}))$, where $C^u(G^{\mathcal{V}})$ is the universal C^*-algebra associated to G.*

2. *for any compact quantum group G acting on $(\mathcal{M}, \mathcal{H}, \mathcal{V}_t)$ in the sense of Definition 3.3.6 with implementing unitary representation $\widehat{U} \in M(K(\mathcal{H}) \otimes C^u(G))$, there exists a surjective*

$*$-homomorphism $C^u(G^{\mathcal{V}}) \to C^u(G)$ which maps $\mathbb{U} \mapsto \widehat{U}$.

We define the quantum isometry group of the W^*-quantum metric space, $G^{\mathcal{V}}$, to be the universal compact quantum group (if such a universal object exists) acting on the quantum metric space as in Definition 3.3.6.

It's not clear whether such a universal object exists in general. However, it can be shown that one exists in the finite dimensional case, which leads us into our next section.

3.3.2 Finite dimensional case

It is known that the quantum metrics do not depend on the choice of Hilbert space on which $\mathcal{M}$ is represented, and this result will be crucial to us.

Theorem 3.3.9. *([27], Theorem 2.4.) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{M}_1 \subseteq B(\mathcal{H}_1)$ and $\mathcal{M}_2 \subseteq B(\mathcal{H}_2)$ be isomorphic von Neumann algebras. Then any isomorphism induces an order preserving 1-1 correspondence between the quantum metrics on $\mathcal{M}_1$ and $\mathcal{M}_2$.*

Therefore, for the following definition we may assume that we're representing $\mathcal{V}_t \subseteq B(L^2(\mathcal{M}))$ in the regular representation of $\mathcal{M}$ where $L^2(\mathcal{M}) = L^2(\mathcal{M}, \psi)$ where ψ is the unique δ-map.

Definition 3.3.10. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a finite dimensional von Neumann algebra with its canonical δ-trace fixed. The quantum automorphism group of $\mathcal{M}$, denoted G_{aut}, is the universal compact quantum group with the following properties:*

1. *$C^u(G_{aut})$ is generated by the entries of a representation $U \in B(\mathcal{H}) \otimes C^u(G_{aut})$*
2. *By identifying $\mathcal{H} = \mathcal{M}$ as vector spaces, then we define the trace-preserving unital $*$-homomorphism δ on $\mathcal{M}$ as*

$$\begin{aligned} \delta : \mathcal{M} &\to \mathcal{M} \otimes C^u(G_{aut}) \\ e_j &\mapsto \sum_k e_k \otimes u_{kj} \end{aligned} \tag{3.2}$$

where $U = [u_{ij}]$ is the fundamental unitary representation.

This U is a unitary representation of the compact quantum group using the natural comultiplication $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ if and only if δ as defined above is an action.

Since we assume that δ is a trace-preserving unital $*$-homomorphism, [5] shows that U is automatically unitary.

Now, for any action $\alpha : \mathcal{M} \to \mathcal{M} \otimes C(G)$ that preserves the canonical trace, it was shown in [41] that for the Hilbert space $\mathcal{H} = L^2(\mathcal{M}) = L^2(\mathcal{M}, \psi)$, there always exists a unitary representation $V \in B(\mathcal{H}) \otimes C(G)$ that implements α via $\alpha(T) = V^*(T \otimes 1)V$ for each $T \in \mathcal{M}$.

Definition 3.3.11. *Let $\mathcal{M}$ be a finite dimensional von Neumann algebra. Fix a W^*-quantum metric $(\mathcal{V}_t)_{t\geq 0}$, where by Theorem 3.3.9 we may assume that the metric space is represented on the GNS Hilbert space, $(\mathcal{V}_t) \subseteq B(L^2(\mathcal{M}))$.*

We define the quantum isometry group $\mathcal{O}(G^{\mathcal{V}})$ to be the quantum subgroup of G_{aut} generated by u_{ij} where the map δ in (3.2) is a ψ-preserving $*$-homomorphism and the conjugation action $\alpha_{\mathcal{V}}$ given by

$$\begin{aligned} \alpha_{\mathcal{V}} : B(L^2(\mathcal{M})) &\to B(L^2(\mathcal{M})) \otimes \mathcal{O}(G^{\mathcal{V}}) \\ T &\mapsto U(T \otimes 1)U^* \end{aligned} \tag{3.3}$$

leaves the $\mathcal{V}_t$ invariant, i.e. $\alpha_{\mathcal{V}}(\mathcal{V}_t) \subseteq \mathcal{V}_t \otimes \mathcal{O}(G^{\mathcal{V}})$ for all t.

Here, $G^{\mathcal{V}}$ will be of Kac type. Indeed, $G^{\mathcal{V}}$ is a quantum subgroup of G_{aut}, the quantum automorphism group of a tracial von Neumann algebra, which is known to be of Kac type.

Proposition 3.3.12. *Let (X, d) be a classical metric space, and consider the construction of the W^*-quantum metric space as in Proposition 3.2.2, with $\mathcal{M} = \ell^\infty(X)$ and $\mathcal{H} = \ell^2(X)$. Then the quantum isometry group of the metric space is the same as the quantum isometry group of the corresponding W^*-quantum metric space, that is, $G^{\mathcal{V}} \cong G^X$.*

Proof. Banica's (G^X, U) can be seen to satisfy the properties of $G^{\mathcal{V}}$ and so $G^X < G^{\mathcal{V}}$.

To see the inclusion $G^{\mathcal{V}} < G^X$, by the properties defining the quantum automorphism group, we know the fundamental representation U of $G^{\mathcal{V}}$ is a magic unitary. Therefore, $G^{\mathcal{V}} < S_n^+$ where $n = |X|$. We need to check that $U(D \otimes 1)U^* = D$ follows from the invariance of $\alpha_{\mathcal{V}}$.

For $x, y \in X$ with $d(x, y) \leq t$, we know that $\alpha_{\mathcal{V}}(V_{xy}) \in \mathcal{V}_t^X$, and so we may write it as $\alpha_{\mathcal{V}}(V_{xy}) = \sum_{d(s,k) \leq t} V_{sk} \otimes x_{sk}$ for some $x_{sk} \in C(G)$. By the definition of $\alpha_{\mathcal{V}}$, we see that $\alpha_{\mathcal{V}}(V_{xy}) = \sum_{s,k} V_{sk} \otimes u_{xs} u_{yk}$. Consider $a, b \in X$ with $d(a, b) > t$. We may start with

$$\sum_{s,k,d(s,k)\leq t} V_{sk} \otimes x_{sk} = \sum_{s,k} V_{sk} \otimes u_{xs} u_{yk}$$

We multiply by $(V_{aa} \otimes 1)$ on the left and by $(V_{bb} \otimes 1)$ on the right to obtain

$$\sum_{s,k,d(s,k)\leq t} V_{aa} V_{sk} V_{bb} \otimes x_{sk} = \sum_{s,k} V_{aa} V_{sk} V_{bb} \otimes u_{xs} u_{yk}. \tag{3.4}$$

We will use the fact that $V_{aa} V_{sk} V_{bb} = \delta_{s=a} \delta_{k=b} V_{ab}$. Thus, the sum on the left hand side of equation (3.4) equals zero since $d(s, k) = d(a, b) > t$. On the right hand side of equation (3.4), the sum simply condenses down to $V_{ab} \otimes u_{xa} u_{yb}$. Therefore

$$0 = V_{ab} \otimes u_{xa} u_{yb}.$$

So indeed, if $d(x, y) \neq d(a, b)$ then $V_{ax} V_{by} = 0$ implying

$$0 = (V_{ax} V_{by})^* = V_{by}^* V_{ax}^* = V_{by} V_{ax}.$$

Similarly, by using the antipode or considering an analogous β-action, it can be shown that $V_{xa} V_{yb} = 0 = V_{yb} V_{xa}$.

We now claim that this is equivalent to $U(D \otimes 1)U^* = D$. Indeed, the (i, j) entry of $U(D \otimes 1)U^*$ can be calculated as follows:

$$
\begin{aligned}
[U(D\otimes 1)U^*]_{ij} &= \sum_{k,s} d(k,s)u_{ik}u_{js} \\
&= \sum_{k,s,d(i,j)=d(k,s)} d(i,j)u_{ik}u_{js} \\
&= d(i,j) \sum_{k,s,d(i,j)=d(k,s)} u_{ik}u_{js} \\
&= d(i,j) \sum_{k,s} u_{ik}u_{js} \\
&= d(i,j) \left(\sum_{k} u_{ik}\right) \left(\sum_{j} u_{js}\right) \\
&= d(i,j)\cdot 1 = d(i,j)
\end{aligned}
$$

So then $G^{\mathcal{V}} < G^X$. $\square$

Remark 3.3.13. *Extensions of Banica's quantum isometry group on classical metric spaces were studied in [35], where quantum isometry groups of quantum metric spaces in the framework of Rieffel is studied. Although both our definition and theirs agrees with Banica's definition in the classical sense, it would be interesting to further investigate the connection between the two extensions.*

4. BIGALOIS EXTENSIONS

4.1 Monoidal equivalence and bigalois extensions

For a CQG G, the *representation category of* G, Rep(G), has a lot of nice structure. In particular it is an example of a so called *strict C*-tensor category with conjugates*. See [33] for more details.

We now come to a notion of central importance in this work: monoidal equivalence of compact quantum groups. Let G be a CQG. Denote by Irr(G) the set of equivalence classes of irreducible objects in Rep(G).

Definition 4.1.1 ([7, 9])**.** *Let G_1, G_2 be two compact quantum groups. We say that G_1 and G_2 are monoidally equivalent, and write $G_1 \sim^{mon} G_2$, if there exists a bijection*

$$\varphi : \mathit{Irr}(G_1) \to \mathit{Irr}(G_2)$$

together with linear isomorphisms

$$\varphi : \mathit{Mor}(u_1 \otimes \ldots \otimes u_n, v_1 \otimes \ldots \otimes v_m) \to \mathit{Mor}(\varphi(u_1) \otimes \ldots \otimes \varphi(u_n), \varphi(v_1) \otimes \ldots \otimes \varphi(v_m))$$

such that $\varphi(1_{G_1}) = 1_{G_2}$ (1_{G_i} being the trivial representation of G_i), and such that for any morphisms S, T,

$$\varphi(S \circ T) = \varphi(S) \circ \varphi(T) \quad (\textit{whenever } S \circ T \textit{ is well-defined})$$
$$\varphi(S^*) = \varphi(S)^*$$
$$\varphi(S \otimes T) = \varphi(S) \otimes \varphi(T).$$

A monoidal equivalence between G_1 and G_2 means that the strict C*-tensor categories Rep(G_i) are *unitarily monoidally equivalent*. More precisely, the maps φ defined above canonically extend to a unitary tensor functor $\varphi : \mathrm{Rep}(G_1) \to \mathrm{Rep}(G_2)$ that is *fully faithful* (i.e., φ defines an isomorphism between $\mathrm{Mor}(u, v)$ and $\mathrm{Mor}(\varphi(u), \varphi(v))$ for any objects $u, v \in \mathrm{Rep}(G_1)$) and is *essentially surjective* (i.e., every object in $\mathrm{Rep}(G_2)$ is of the form $\varphi(u)$ for some $u \in \mathrm{Rep}(G_i)$).

We now discuss an equivalent, but somewhat more concrete, way to think about monoidal equivalence of compact quantum groups. The key object is that of a bigalois extension, which has its origins in Hopf algebra theory, but is adapted here to the analytic setting of CQGs.

Let $A = \mathcal{O}(G)$ be a Hopf $*$-algebra of representative functions on a CQG G. A *left A $*$-comodule algebra* is a unital $*$-algebra Z equipped with a unital $*$-homomorphism $\alpha : Z \to A \otimes Z$ satisfying $(\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha$ and $(\epsilon \otimes \iota)\alpha = \iota$. Similarly, a *right A $*$-comodule algebra* is a unital $*$-algebra Z equipped with a unital $*$-homomorphism $\beta : Z \to Z \otimes A$ satisfying $(\beta \otimes \iota)\beta = (\iota \otimes \Delta)\beta$ and $(\iota \otimes \epsilon)\beta = \iota$.

A left A $*$-comodule algebra (Z, α) is called a *left A Galois extension* if the linear map

$$\kappa_l : Z \otimes Z \to A \otimes Z; \qquad \kappa_l(x \otimes y) = \alpha(x)(1 \otimes y)$$

is bijective. Similarly, a right A $*$-comodule algebra (Z, β) is called a *right A Galois extension* if the linear map

$$\kappa_r : Z \otimes Z \to Z \otimes A; \qquad \kappa_r(x \otimes y) = (x \otimes 1)\beta(y)$$

is bijective. Finally, let A and B be Hopf $*$-algebras. A unital $*$-algebra Z is called an *$A - B$ bigalois extension* if it is both a left A Galois extension and a right B Galois extension, and Z is an $A - B$-bicomodule algebra. I.e., if α, β denote the left and right comodule maps, respectively, then we have the equality of maps

$$(\iota \otimes \beta)\alpha = (\alpha \otimes \iota)\beta : Z \to A \otimes Z \otimes B.$$

Remark 4.1.2. *The notion of a (bi)galois extension should be regarded as a quantum analogue of the familiar notion of a (bi)torsor (or principle homogeneous (bi)bundle) in the context of group* *actions: If G is a (finite) group and $G \curvearrowright X$ is an action of G on a finite space X, we call X a (left) G-torsor if the action is free and transitive. This is equivalent to saying that the canonical map*

$$G \times X \to X \times X; \qquad (g,t) \mapsto (g \cdot t, t)$$

is bijective. Letting $\mathcal{O}(X)$ denote the $*$-algebra of functions on X, then $\mathcal{O}(X)$ is a left $\mathcal{O}(G)$ $*$-comodule algebra with the map

$$\begin{aligned} &\alpha : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X) \cong \mathcal{O}(G \times X) \\ &\alpha(f)(g,t) = f(g \cdot t). \end{aligned}$$

With these definitions, it is clear that $G \curvearrowright X$ is free and transitive if and only if

$$\begin{aligned} \kappa_l &: \mathcal{O}(X) \otimes \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X) \\ &\kappa_l(x \otimes y)(g,t) = \alpha(x)(1 \otimes y)(g,t) = x(g \cdot t)y(t) \end{aligned}$$

is bijective, i.e., if and only if $\mathcal{O}(X)$ is a left $\mathcal{O}(G)$-galois extension. Similar statements hold for right G-spaces and left-right G_1-G_2-spaces.

In the following, we will be interested in bigalois extensions which admit non-zero C*-envelopes. The main way in which this is achieved is by considering necessary and sufficient conditions for the existence of invariant states on bigalois extensions. In what follows, a *state* on a unital $*$-algebra Z is a linear functional $\omega : Z \to \mathbb{C}$ such that $\omega(1) = 1$ and $\omega(z^*z) \geq 0$ for all $z \in Z$.

Definition 4.1.3. *Let Z be an $A-B$-bigalois extension. A state $\omega : Z \to \mathbb{C}$ is called left-invariant if $(\iota \otimes \omega)(z) = \omega(z)1_A$ for each $z \in Z$, and it is called right-invariant if $(\omega \otimes \iota)(z) = \omega(z)1_B$ for each $z \in Z$. We call ω bi-invariant if it is both left and right-invariant.*

Example 4.1.4. *The Hopf $*$-algebra $A = \mathcal{O}(G)$ of representative functions on a compact quantum* *group G is a natural example of an $A-A$-bigalois extension admitting a bi-invariant state. Indeed, just take $\omega = h$, the Haar state on A.*

The following theorem summarizes some useful properties of bi-invariant states on bigalois extensions. It is an amalgamation of various results in [7, 9, 37].

Theorem 4.1.5. *Let G_1, G_2 be compact quantum groups with $A = \mathcal{O}(G_1)$ and $B = \mathcal{O}(G_2)$. Let Z be an $A - B$-bigalois extension. Then we have the following.*

1. *Any left/right/bi-invariant state $\omega : Z \to \mathbb{C}$ is unique and faithful (if it exists).*

2. *The following are equivalent:*

   (a) *Z admits a non-zero $*$-representation as bounded linear operators on a Hilbert space.*

   (b) *Z admits a state.*

   (c) *Z admits a bi-invariant state.*

   (d) *Z admits a left (resp. right)-invariant state.*

3. *If Z admits a bi-invariant state ω, denote by $B^u(G_1, G_2) \neq 0$ the universal **C***-algebra generated by Z and by $B_r = \overline{\pi_\omega(Z)}$, the **C***-algebra generated by the GNS representation with respect to ω. Then ω extends to a KMS state on both B^u and B_r. Moreover, ω is a tracial state if and only if both G_1 and G_2 are of Kac type. (I.e., the Haar states on both $\mathcal{O}(G_i)$ are tracial)*

Theorem 4.1.6 ([7, 9])**.** *Let G_1, G_2 be compact quantum groups. Then G_1 and G_2 are monoidally equivalent if and only if there exists an $\mathcal{O}(G_1) - \mathcal{O}(G_2)$-bigalois extension Z equipped with a bi-invariant state ω.*

We refer the reader to [33, Theorem 2.3.11] or [9, Theorem 3.9 and Proposition 3.13] for a precise description of the the bigalois extension (Z, ω) induced by the monoidal equivalence in Theorem 4.1.6.

We end this section by stating a simple criterion due to Bichon [7] (for compact matrix quantum groups) for a bigalois extension to admit an invariant state ω. First we need some definitions. Let $n \in \mathbb{N}$ and $F_i \in \mathrm{GL}_n(\mathbb{C})$. We define $\mathcal{O}(U_{F_1}^+, U_{F_2}^+)$ to be the unital $*$-algebra generated by the coefficients z_{ij} of a $n_1 \times n_2$ matrix $z = [z_{ij}]_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} \in M_{n_1,n_2}(\mathcal{O}(U_{F_1}^+, U_{F_2}^+))$ satisfying the relations making both z and $F_1\bar{z}F_2^{-1}$ unitary, where $\bar{z} = [z_{ij}^*]$. When $F_1 = F_2 = F$, note that $\mathcal{O}(U_F^+, U_F^+) = \mathcal{O}(U_F^+)$ is the Hopf $*$-algebra of representative functions on the universal unitary quantum group U_F^+ introduced earlier. We also note that if $\mathcal{O}(U_{F_1}^+, U_{F_2}^+) \neq 0$ then $\mathcal{O}(U_{F_1}^+, U_{F_2}^+)$ is an $\mathcal{O}(U_{F_1}^+) - \mathcal{O}(U_{F_2}^+)$-bigalois extension with respect to the bicomodule structure given by

$$\alpha_{F_1,F_2} : \mathcal{O}(U_{F_1}^+, U_{F_2}^+) \to \mathcal{O}(U_{F_1}^+) \otimes \mathcal{O}(U_{F_1}^+, U_{F_2}^+); \qquad \alpha_{F_1,F_2}(z_{ij}) = \sum_{k=1}^{n_1} u_{ik} \otimes z_{kj}$$

$$\beta_{F_1,F_2} : \mathcal{O}(U_{F_1}^+, U_{F_2}^+) \to \mathcal{O}(U_{F_1}^+, U_{F_2}^+) \otimes \mathcal{O}(U_{F_2}^+); \qquad \beta_{F_1,F_2}(z_{ij}) = \sum_{l=1}^{n_2} z_{il} \otimes v_{lj},$$

where $u = [u_{ij}], v = [v_{ij}]$ are the fundamental representations of $U_{F_1}^+, U_{F_2}^+$, respectively.

Theorem 4.1.7 (Proposition 6.2.6 in [7])**.** *Let G be a compact matrix quantum group and (Z, α) a left $\mathcal{O}(G)$-Galois extension. Let $F \in GL_n(\mathbb{C})$ be such that $G < U_F^+$ (with corresponding surjective morphism $\pi : \mathcal{O}(U_F^+) \to \mathcal{O}(G)$). If there exists $F_1 \in GL_{n_1}(\mathbb{C})$ and a surjective $*$-homomorphism $\sigma : \mathcal{O}(U_F^+, U_{F_1}^+) \to Z$ satisfying $\alpha \circ \sigma = (\pi \otimes \sigma)\alpha_{F,F_1}$, then Z admits a left-invariant state $\omega : Z \to \mathbb{C}$.*

4.2 Bigalois extensions and quantum isomorphisms of graphs [1]

The aim of this section is to show that a quantum isomorphism between two graphs X and Y is nothing other than a (quotient of a) $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-bigalois extension in disguise. We begin by extending the definition of the graph isomorphism game $*$-algebra $\mathcal{A}(Iso(X, Y))$ to include quantum graphs.

[1] Reprinted with permission from "Bigalois extensions and the graph isomorphism game" by M. Brannan, A. Chirvasitu, K. Eifler, S. Harris, V. Paulsen, X. Su, and M. Wasilewski, *Communications in Mathematical Physics*, 375:1777-1809, 2020.

Definition 4.2.1. *Let $X = (\mathcal{O}(X), \psi_X, A_X)$ and $Y = (\mathcal{O}(Y), \psi_Y, A_Y)$ be quantum graphs with $|X| = n$ and $|Y| = m$, and fix orthonormal bases $\{e_j\}$ and $\{f_i\}$ for $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ relative to ψ_X and ψ_Y respectively. Let $\mathcal{O}(G_Y, G_X)$ be the universal $*$-algebra generated by the entries p_{ij} of a unitary matrix*

$$p = [p_{ij}]_{ij} \in \mathcal{O}(G_Y, G_X) \otimes B(L^2(X), L^2(Y))$$

with relations ensuring that

$$\rho_{Y,X} : \mathcal{O}(X) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y, G_X); \qquad e_j \mapsto \sum_i f_i \otimes p_{ij}$$

is a unital $*$-homomorphism satisfying

$$\rho_{Y,X}(A_X \cdot) = (A_Y \otimes \iota)\rho_{Y,X}. \tag{4.1}$$

Our first observation is that the above morphism $\rho_{Y,X}$, if it exists, is automatically state-preserving.

Lemma 4.2.2. *Assume $\mathcal{O}(G_Y, G_X) \neq 0$. Then the morphism $\rho_{Y,X} : \mathcal{O}(X) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y, G_X)$ is state-preserving in the sense that*

$$(\psi_Y \otimes \iota)\rho_{Y,X} = \psi_X(\cdot)1 : \mathcal{O}(X) \to \mathcal{O}(G_Y, G_X). \tag{4.2}$$

Proof. Consider the matrix $p = [p_{ij}]_{ij} \in \mathcal{O}(G_Y, G_X) \otimes B(L^2(X), L^2(Y))$, viewed canonically as a linear map

$$p : L^2(X) \otimes \mathcal{O}(G_Y, G_X) \to L^2(Y) \otimes \mathcal{O}(G_Y, G_X); \qquad p(\xi \otimes a) = \sum_{i,j} |f_i\rangle\langle e_i|\xi\rangle \otimes p_{ij}a.$$

It then follows that $\rho_{Y,X}(\xi) = p(\xi \otimes 1)$ for each $\xi \in L^2(X)$ (here and below we are identifying $L^2(X)$ and $L^2(Y)$ with $\mathcal{O}(X)$ and $\mathcal{O}(Y)$). Consider now the $\mathcal{O}(G_Y, G_X)$-valued sesquilinear forms on $L^2(X) \otimes \mathcal{O}(G_Y, G_X)$ and $L^2(Y) \otimes \mathcal{O}(G_Y, G_X)$ given by

$$\langle \xi_1 \otimes a | \xi_2 \otimes b \rangle_{L^2(X)\otimes\mathcal{O}(G_Y,G_X)} = b^* a \psi_X(\xi_2^* \xi_1) \quad \& \quad \langle \eta_1 \otimes a | \eta_2 \otimes b \rangle_{L^2(Y)\otimes\mathcal{O}(G_Y,G_X)} = b^* a \psi_Y(\eta_2^* \eta_1).$$

Then a simple calculation using the fact that $p^*p = 1$ and $p(1 \otimes 1) = \rho_{Y,X}(1) = 1 \otimes 1$ gives

$$\begin{aligned}
(\psi_Y \otimes \iota)\rho_{Y,X}(\xi) &= (\psi_Y \otimes \iota)p(\xi \otimes 1) \\
&= \langle p(\xi \otimes 1) | 1 \otimes 1 \rangle_{L^2(Y)\otimes\mathcal{O}(G_Y,G_X)} \\
&= \langle p(\xi \otimes 1) | p(1 \otimes 1) \rangle_{L^2(Y)\otimes\mathcal{O}(G_Y,G_X)} \\
&= \langle p^* p(\xi \otimes 1) | 1 \otimes 1 \rangle_{L^2(X)\otimes\mathcal{O}(G_Y,G_X)} \\
&= \langle (\xi \otimes 1) | 1 \otimes 1 \rangle_{L^2(X)\otimes\mathcal{O}(G_Y,G_X)} \\
&= \psi_X(\xi)1.
\end{aligned}$$

□

Remark 4.2.3. *When the two quantum graphs coincide we have $\mathcal{O}(G_X, G_X) = \mathcal{O}(G_X)$ (the Hopf $*$-algebra of polynomial functions on the quantum automorphism group G_X) and $\rho_{X,X} = \rho_X$. For classical graphs X, Y, we have $\mathcal{O}(G_Y, G_X) = \mathcal{A}(Iso(Y, X))$. Indeed, the fact that $\rho_{Y,X}$ is a unital $*$-homomorphism intertwining the quantum adjacency matrices A_X and A_Y says exactly that the unitary matrix $p = [p_{ij}]$ satisfies $(1 \otimes A_Y)p = p(1 \otimes A_X)$ and has entries which are self-adjoint projections satisfying $\sum_i p_{ji} = 1 = \sum_j p_{ji}$, $p_{ji}p_{jl} = \delta_{il}p_{ji}$, and $p_{ij}p_{lj} = \delta_i p_{ij}$. Compare with Proposition 2.3.1 and Remark 2.3.2. See also [2, 30].*

With the above in mind, we now provide a natural extension of the notion of quantum isomorphism to our quantum graphs. Compare with [31].

Definition 4.2.4. *Let X, Y be quantum graphs. We say that X is algebraically quantum isomorphic to Y if $\mathcal{O}(G_Y, G_X) \neq 0$, and write $X \cong_{A^*} Y$. If $\mathcal{O}(G_Y, G_X)$ admits a non-zero* **C****-representation, then we say that X is* **C****-algebraically quantum isomorphic to Y, and write $X \cong_{A^*} Y$. Finally,* *we say $X \cong_{qc} Y$ if $\mathcal{O}(G_Y, G_X)$ admits a tracial state (following the existent notation for classical graphs).*

For the remainder of the present discussion we fix two quantum graphs X, Y as above and assume that the $*$-algebra $\mathcal{O}(G_Y, G_X)$ is non-zero. Our aim is to show that $\mathcal{O}(G_Y, G_X)$ admits a natural structure as a $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$ bigalois extension.

Consider the comodule-algebra structure map

$$\rho_Y : \mathcal{O}(Y) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y).$$

By the universality of $\rho_{Y,X}$, the composition

$$(\rho_Y \otimes \mathrm{id}) \circ \rho_{Y,X} : \mathcal{O}(X) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y) \otimes \mathcal{O}(G_Y, G_X)$$

must factor as $(\mathrm{id} \otimes \alpha) \circ \rho_{Y,X}$ for a unique $*$-algebra morphism

$$\alpha : \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y) \otimes \mathcal{O}(G_Y, G_X)$$

given simply by

$$\alpha(p_{ij}) = \sum_k u_{ik} \otimes p_{kj},$$

where $u = [u_{ij}]$ is the fundamental representation of $\mathcal{O}(G_Y)$.

Similarly, $\mathcal{O}(G_Y, G_X)$ has a right $\mathcal{O}(G_X)$ $*$-comodule algebra structure given by

$$\beta : \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_X); \qquad \beta(p_{ij}) = \sum_k p_{ik} \otimes v_{kj},$$

where $v = [v_{ij}]$ is the fundamental representation of $\mathcal{O}(G_X)$. It is also clear that $\mathcal{O}(G_Y, G_X)$ is an $\mathcal{O}(G_Y) - \mathcal{O}(G_X)$ bicomodule with respect to α and β.

Continuing in the same vein, we can define "cocomposition" $*$-morphisms

$$\gamma_Y : \mathcal{O}(G_Y) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_X, G_Y); \qquad \gamma_Y(u_{ij}) = \sum_k p_{ik} \otimes q_{ki}$$
$$\gamma_X : \mathcal{O}(G_X) \to \mathcal{O}(G_X, G_Y) \otimes \mathcal{O}(G_Y, G_X); \qquad \gamma_X(v_{ij}) = \sum_k q_{ik} \otimes p_{kj}$$

where $q = [q_{ij}]$ is the matrix of generators of $\mathcal{O}(G_X, G_Y)$. For example, to construct γ_Y, we consider the morphism

$$(\rho_{Y,X} \otimes \iota)\rho_{X,Y} : \mathcal{O}(Y) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_X, G_Y).$$

By universality of ρ_Y, there exists a unique morphism $\gamma_Y : \mathcal{O}(G_Y) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_X, G_Y)$ so that

$$(\rho_{Y,X} \otimes \iota)\rho_{X,Y} = (\iota \otimes \gamma_Y)\rho_{X,Y}.$$

This map is readily seen to be given by the proposed formula above.

Thus far, the algebras $\mathcal{O}(G_X)$, $\mathcal{O}(G_Y)$, $\mathcal{O}(G_Y, G_X)$ and $\mathcal{O}(G_X, G_Y)$ together with the maps α and β, their analogues for $\mathcal{O}(G_X, G_Y)$, and γ_X, γ_Y constitute a two-object *cocategory* $\mathcal{C}$ in the sense of [8, Definition 2.1]: the four algebras are to be thought of as dual to "spaces of morphisms" between two objects ($x \to x$ for $\mathcal{O}(G_X)$, $x \to y$ for $\mathcal{O}(G_Y, G_X)$, etc.), and the γ maps are dual to morphism composition.

Next, we make $\mathcal{C}$ into a *cogroupoid* in the sense of [8, Definitions 2.3 and 2.4]: this entails defining "coinversion" maps

$$S_{X,Y} : \mathcal{O}(G_X, G_Y) \to \mathcal{O}(G_Y, G_X) \tag{4.3}$$
$$S_{Y,X} : \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_X, G_Y), \tag{4.4}$$

which will require some preparation.

Let $F = F_X \in M_n$ and $G = F_Y \in M_m$ be matrices with the property that $Fe_i = e_i^*$ and similarly for G, so that $\overline{F} = F^{-1}$ and $\overline{G} = G^{-1}$. Note that the involutivity of the morphisms

$$\rho_X : \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G_X)$$

$$\rho_Y : \mathcal{O}(Y) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y)$$

$$\rho_{Y,X} : \mathcal{O}(X) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y, G_X)$$

is equivalent, respectively, to the equalities

$$(1 \otimes F)\bar{u} = u(1 \otimes F) \tag{4.5}$$

$$(1 \otimes G)\bar{v} = v(1 \otimes G) \tag{4.6}$$

$$(1 \otimes G)\bar{p} = p(1 \otimes F) \tag{4.7}$$

We will henceforth abuse notation and write uF for $u(1 \otimes F)$, etc. Taking this into account, we have

$$G^{-1}pF = \overline{p} \text{ and similarly } F^{-1}qG = \overline{q}.$$

It is now a simple check to see that

$$f_i \mapsto \sum_j e_j \otimes p_{ij}^* \tag{4.8}$$

defines a unital algebra homomorphism

$$\mathcal{O}(X) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y, G_X)^{op}. \tag{4.9}$$

Applying G to both sides of (4.8), writing $e_j = FF^{-1}e_j$ and using

$$Fe_j = e_j^*, \quad Gf_i = f_i^*,$$

it follows that (4.8) is involutive with respect to the modified $*$-structure $\star$ on $\mathcal{O}(G_Y, G_X)^{op}$ given by

$$(p^*)^\star = (F^{-1} p^* G)^t$$

(the 't' superscript denoting the transpose). The defining universality property of $\mathcal{O}(G_X, G_Y)$ then implies that the morphism (4.9) given by (4.8) factors as

$$(\iota \otimes S_{X,Y})\rho_X$$

for a conjugate-linear anti-morphism (4.3). $S_{Y,X}$ is defined similarly, and in summary we have

$$\begin{aligned} S_{X,Y} : &\quad q \mapsto p^*, \quad q^* \mapsto G^t \overline{p} F^{-t} \\ S_{Y,X} : &\quad p \mapsto q^*, \quad p^* \mapsto F^t \overline{p} G^{-t} \end{aligned}$$

where the 't' superscript means 'transpose' while '$-t$' denotes 'transpose inverse'.

The morphisms (4.3) and (4.4) enrich the above-mentioned cocategory $\mathcal{C}$ to a connected *cogroupoid* in the sense of [8, Definitions 2.3 and 2.4].

We are now ready for the main result of this section.

Theorem 4.2.5. *If $\mathcal{O}(G_Y, G_X)$ is non-zero, then $(\mathcal{O}(G_Y, G_X), \alpha, \beta)$ is a $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-bigalois extension.*

Proof. By [8, Proposition 2.8] this is an immediate consequence of $\mathcal{C}$ being a connected cogroupoid. More precisely, the arguments therein show that the relevant linear maps

$$\kappa_l : \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y) \otimes \mathcal{O}(G_Y, G_X)$$
$$\kappa_l(x \otimes y) = \alpha(x)(1 \otimes y)$$
$$\kappa_r : \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_X)$$
$$\kappa_r(x \otimes y) = (x \otimes 1)\beta(y)$$

are bijective with explicit inverses given by

$$\eta_l : \mathcal{O}(G_Y) \otimes \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_Y, G_X)$$
$$\eta_l = (\iota \otimes m)(\iota \otimes S_{X,Y} \otimes \iota)(\gamma_Y \otimes \iota)$$
$$\eta_r : \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_X) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_Y, G_X)$$
$$\eta_r = (m \otimes \iota)(\iota \otimes S_{X,Y} \otimes \iota)(\iota \otimes \gamma_X)$$

where m denotes the multiplication map in the appropriate algebra. □

Theorem 4.2.5 puts some of the material in [30] in a category-theoretic perspective. To make sense of this, we need to recall

Definition 4.2.6. *The quantum orbital algebra of a (quantum) graph X is the endomorphism algebra of $\mathcal{O}(X)$ as a comodule over $\mathcal{O}(G_X)$. That is, the algebra of intertwiners Mor$(u, u) \subset B(L^2(X))$, where u denotes the fundamental representation of G_X.*

In the case of classical graphs, this is not quite [30, Definition 3.10], but is equivalent to it by [30, Theorem 3.11]. Note that [30, Theorem 4.2] follows from Theorem 4.2.5: the former says that a quantum isomorphism between two (classical) graphs entails an isomorphism between their quantum orbital algebras that identifies the respective adjacency matrices. Since by Theorem 4.2.5

we have a category equivalence

$$\mathrm{Rep}(G_X) \simeq \mathrm{Rep}(G_Y)$$

identifying $\mathcal{O}(X)$ on the left to $\mathcal{O}(Y)$ on the right, this implements an isomorphism between the endomorphism algebras of these two objects in the respective categories (i.e. the quantum orbital algebras). Furthermore, the fact that this isomorphism identifies A_X and A_Y follows from (4.1).

4.2.1 Existence of states on $\mathcal{O}(G_Y, G_X)$

Our next result shows that $\mathcal{O}(G_Y, G_X)$ always admits a faithful bi-invariant state (and hence a C*-completion) whenever this algebra is non-zero.

Theorem 4.2.7. *Let X, Y be quantum graphs. If $\mathcal{O}(G_Y, G_X) \neq 0$, then there exists a faithful bi-invariant state $\omega : \mathcal{O}(G_Y, G_X) \to \mathbb{C}$, and therefore we have a monoidal equivalence of compact quantum groups $G_X \sim^{mon} G_Y$. Moreover, ω is tracial if and only if both G_X and G_Y are of Kac type.*

Proof. Recall the matrices $F = F_X$ and $G = F_Y$ from the preceding discussion. The equations (4.6) imply that we have surjective $*$-homomorphisms $\pi : \mathcal{O}(U^+_{F_Y}) \to \mathcal{O}(G_Y)$ and $\sigma : \mathcal{O}(U^+_{F_Y}, U^+_{F_X}) \to \mathcal{O}(G_Y, G_X)$ satisfying $\alpha \circ \sigma = (\pi \otimes \sigma)\alpha_{F_Y, F_X}$. By Theorems 4.1.5 and 4.1.7 $\mathcal{O}(G_Y, G_X)$ then admits a $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-invariant state (which is tracial precisely when G_Y, G_X are both of Kac type). By Theorem 4.1.6, $G_X \sim^{\mathrm{mon}} G_Y$. $\square$

Corollary 4.2.8. *Let X and Y be quantum graphs. Then the following are equivalent.*

(1) $X \cong_{A^*} Y$.

(2) $X \cong_{C^*} Y$.

Moreover, if both X and Y are equipped with tracial δ-forms, then $X \cong_{qc} Y$.

Proof. $(2) \implies (1)$ by definition, while the converse follows from Theorem 4.2.7. The same theorem also shows that when G_X and G_Y are Kac (as is the case if X and Y are equipped with tracial δ-forms) $\mathcal{O}(G_Y, G_X)$ is equipped with a trace. This proves the last claim. $\square$

Restricting our attention to classical graphs X, and Y we arrive at one of the main results of the paper.

Theorem 4.2.9. *Let X and Y be classical graphs. Then the following conditions are equivalent.*

1. $X \cong_{A^*} Y$.

2. $X \cong_{qc} Y$.

3. $X \cong_{C^*} Y$.

Proof. This is an immediate consequence of Corollary 4.2.8. □

Remark 4.2.10. *The above theorems show that the algebras $\mathcal{O}(G_Y, G_X)$ are non-zero in the category of $*$-algebras if and only if $\mathcal{O}(G_Y, G_X)$ admits a non-zero representation as bounded operators on Hilbert space. In other words, the $*$-algebra and C*-algebra worlds coincide for this class of examples.*

One illustration of the distinction between $*$-algebras and C-algebras is in the behavior of projections (i.e. self-adjoint idempotents). In a C*-algebra, if one has self-adjoint idempotents $\{p_1, ..., p_N\}$ satisfying $p_1 + \cdots + p_N = 1$, then necessarily $p_i p_j = 0$, $\forall i \neq j$.*

The situation is very different for $*$-algebras, however. While triples of projections with sum 1 still commute, quadruples need not. This can be seen, for instance, from [6]. There, the ring generated by three idempotents a, b and c whose sum is also idempotent ($d = 1 - (a + b + c)$ thus being idempotent as well) is shown to have a basis as a free abelian group consisting of those monomials in a, b and c such that

- *no letter appears twice in succession;*

- *b never appears to the left of a.*

This makes it clear that $ab \neq 0$. One can simply reprise this example over $\mathbb{C}$ (i.e. working with complex algebras rather than rings) and superimpose a $*$-structure by requiring that a, b and c be *self-adjoint. The result is a complex $*$-algebra with four non-orthogonal projections adding up to 1.*

In fact, even more pathological examples exist. In [24] a machine-assisted proof is given that the $*$-algebra $\mathcal{A}(Hom(K_5, K_4))$ is non-trivial. This is a $*$-algebra with generators

$$\{e_{x,a} : 1 \leq x \leq 5, 1 \leq a \leq 4\}$$

satisfying the usual relations, $e_{x,a}^* = e_{x,a}^2 = e_{x,a}$, $e_{x,a}e_{x,b} = 0$, when $a \neq b$, $\sum_{a=1}^4 e_{x,a} = 1$, $\forall x$, and the relations, $e_{x,a}e_{y,a} = 0$, $x \neq y$, prescribed by the graphs.

If one sets $p_a = \sum_x e_{x,a}$, then $p_a^2 = p_a = p_a^*$, for $1 \leq a \leq 4$. Hence, $q_a = 1 - p_a$ are also self-adjoint idempotents. However,

$$\sum_{a=1}^{4} q_a = 4 \cdot 1 - \sum_{a=1}^{4} p_a = 4 \cdot 1 - \sum_{x=1}^{5} \sum_{a=1}^{4} e_{x,a} = 4 \cdot 1 - 5 \cdot 1 = -1.$$

Thus, it is possible to have 4 self-adjoint idempotents sum to -1 in a $*$-algebra.

4.2.2 From monoidal equivalence to quantum isomorphism

Theorem 4.2.7 and corollary 4.2.8 show that for a pair of quantum graphs X, Y the condition $X \cong_{A^*} Y$ implies that the corresponding quantum automorphism groups G_X and G_Y are monoidally equivalent. Based on this connection between quantum isomorphism and monoidal equivalence, it is natural to ask whether the converse holds, namely: does $G_X \sim^{\text{mon}} G_Y$ imply $X \cong_{A^*} Y$?

The answer to this question turns out to be 'no' in general. For example, take $X = K_n$ and $Y = \overline{K_n}$. In this case we have $G_X = G_Y = S_n^+$ (so G_X and G_Y are in particular monoidally equivalent as compact quantum groups), but it is clear from the definitions that $\mathcal{A}(Iso(X,Y)) = \mathcal{O}(G_X, G_Y) = 0$. The intuitive reason for this is that the trivial monoidal equivalence taking $\text{Rep}(S_n^+)$ to itself does not map the adjacency matrix A_X to A_Y. In fact, *there cannot exist any* any monoidal equivalence $\varphi : \text{Rep}(S_n^+) \to \text{Rep}(S_n^+)$ satisfying $\varphi(A_X) = A_Y$. This is because such a monoidal equivalence would force A_X and $\varphi(A_X) = A_Y$ to be isospectral.

On the other hand, the following theorem shows that whenever we have a quantum group G monoidally equivalent to G_X, it is possible to find a quantum graph Y so that $G = G_Y$ and $X \cong_{A^*} Y$.

Theorem 4.2.11. *Let $X = (\mathcal{O}(X), \psi_X, A_X)$ be a quantum graph and G_X its quantum automorphism group. Let G be another compact quantum group that is monoidally equivalent to G_X. Then there exists a quantum graph $Y = (\mathcal{O}(Y), \psi_Y, A_Y)$ so that $G = G_Y$, and we have a quantum isomorphism $X \cong_{A^*} Y$.*

Proof. When X is a quantum complete graph, this result is already known [37, Theorem 3.6.5]. The proof in the case of arbitrary X follows almost verbatim, so we just sketch the main ideas.

Let $\varphi : \text{Rep}(G_X) \to \text{Rep}(G)$ be the unitary fiber functor implementing the monoidal equivalence as in Definition 4.1.1. Put $L^2(Y) = \varphi(L^2(X))$, $d_Y = \dim(L^2(Y))$ and let $v = \varphi(u) \in M_{d_Y}(\mathcal{O}(G))$ be the corresponding unitary representation of G on $L^2(Y)$. Put $m_Y = \varphi(m_X) \in \text{Mor}(v \otimes v, v)$, $\eta_Y = \varphi(\eta_X) \in \text{Mor}(1, v)$ and $\psi_Y = \eta_Y^* \in \text{Mor}(v, 1)$ and $A_Y = \varphi(A_X) \in \text{Mor}(v, v)$. Then exactly as in the proof of [37, Theorem 3.6.5], $L^2(Y)$ is a unital C*-algebra with multiplication m_Y, unit η_Y, involution $\sharp : \xi \mapsto \xi^\sharp = (\iota \otimes \xi^*)(m_Y^* \eta_Y)$, and $\psi_Y : L^2(Y) \to \mathbb{C}$ is a δ-form. We denote this C*-algebra by $\mathcal{O}(Y)$. Finally, consider the map $A_Y : L^2(Y) \to L^2(Y)$. Then by definition of φ, we have

$$
\begin{aligned}
&A_Y^* = \varphi(A_X)^* = \varphi(A_X^*) = \varphi(A_X) = A_Y,\\
&m_Y(A_Y \otimes A_Y)m_Y^* = \varphi(m_X(A_X \otimes A_X)m_X^*) = \varphi(\delta^2 A_X) = \delta^2 A_Y,\\
&(\iota \otimes \eta_Y^* m_Y)(\iota \otimes A_Y \otimes \iota)(m_Y^* \eta_Y \otimes \iota) = \varphi((\iota \otimes \eta_X^* m_X)(\iota \otimes A_X \otimes \iota)(m_X^* \eta_X \otimes \iota)) = \varphi(A_X) = A_Y\\
&m_Y(A_Y \otimes \iota)m_Y^* = \varphi(m_X(A_X \otimes \iota)m_X^*) = \varphi(\delta^2 \iota) = \delta^2 \iota,
\end{aligned}
$$

so A_Y is a quantum adjacency matrix and $Y = (\mathcal{O}(Y), \psi_Y, A_Y)$ is a quantum graph.

Now let G_Y be the quantum automorphism group of Y, with fundamental representation $w \in M_{d_Y}(\mathcal{O}(G_Y))$. Then by Definition 3.1.4 and the construction of the morphisms

$m_Y, \eta_Y, \varphi_Y, A_Y$ using the monoidal equivalence φ, there is a surjective Hopf $*$-homomorphism $\sigma : \mathcal{O}(G_Y) \to \mathcal{O}(G)$ given by $(\sigma \otimes \iota)w = v$. In particular, $G < G_Y$ is a quantum subgroup, which implies that for any $m, n \in \mathbb{N}_0$, we have $\mathrm{Mor}(w^{\otimes m}, w^{\otimes n}) \subseteq \mathrm{Mor}(v^{\otimes m}, v^{\otimes n})$. To prove that in fact $G = G_Y$, it suffices to check equality in the above containments for each m, n (see for example [11, Proposition 3.5]). To this end, recall that by our monoidal equivalence, we have isomorphisms $\varphi : \mathrm{Mor}(u^{\otimes m}, u^{\otimes n}) \cong \mathrm{Mor}(v^{\otimes m}, v^{\otimes n})$. Moreover, since (by universality of G_X) the space $\mathrm{Mor}(u^{\otimes m}, u^{\otimes n})$ is generated (in the C*-tensor categorical sense) by the maps $\{\iota, m_X, \eta_X, A_X\}$, it follows that $\mathrm{Mor}(v^{\otimes m}, v^{\otimes n})$ is also generated the images $\{\varphi(\iota), \varphi(m_X), \varphi(\eta_X), \varphi(A_X)\} = \{\iota, m_Y, \eta_Y, A_Y\}$. But by the same universal reasoning, $\mathrm{Mor}(w^{\otimes m}, w^{\otimes n})$ is generated by $\{\iota, m_Y, \eta_Y, A_Y\}$, so we conclude that $\mathrm{Mor}(v^{\otimes m}, v^{\otimes n}) = \varphi(\mathrm{Mor}(u^{\otimes m}, u^{\otimes n})) \subseteq \mathrm{Mor}(w^{\otimes m}, w^{\otimes n})$.

Finally, it remains to show that $X \cong_{A^*} Y$. Since we have a monoidal equivalence $\varphi : \mathrm{Rep}(G_X) \to \mathrm{Rep}(G_Y)$, Theorem 4.1.6 guarantees the existence of an $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-bigalois extension Z. Moreover, from [33, Theorem 2.3.11], one can construct a unitary operator $z \in Z \otimes B(L^2(X), L^2(Y))$ satisfying the relations

1. $(1 \otimes A_Y)z = (1 \otimes \varphi(A_X))z = z(1 \otimes A_X)$.

2. $1 \otimes \eta_Y = 1 \otimes \varphi(\eta_X) = z(1 \otimes \eta_X)$.

3. $(1 \otimes m_Y)z_{12}z_{13} = (1 \otimes \varphi(m_X))z_{12}z_{13} = z(1 \otimes m_X)$.

4. $(z^*)_{12}(1 \otimes m_Y^*\eta_Y \otimes 1) = (z^*)_{12}(1 \otimes \varphi(m_X^*\eta_X) \otimes 1) = z_{13}(1 \otimes m_X^*\eta_X)$.

These four relations say precisely that the map $e_i \mapsto \sum_j f_j \otimes z_{ji}$ defines a unital $*$-homomorphism $\mathcal{O}(X) \to \mathcal{O}(Y) \otimes Z$ (where (e_i) and (f_j) are ONBs for $L^2(X)$ and $L^2(Y)$). In particular, we obtain a non-zero $*$-homomorphism $\mathcal{O}(G_Y, G_X) \to Z$ given by $p \mapsto z$ (where p denotes the matrix of generators of $\mathcal{O}(G_Y, G_X)$). I.e., $\mathcal{O}(G_Y, G_X) \neq 0$. □

Remark 4.2.12. *With a little more work one can show that in fact $\mathcal{O}(G_Y, G_X) \cong Z$ via the above homomorphism.*

Theorem 4.2.11 supplies us with many easy examples of quantum isomorphic quantum graphs.

Example 4.2.13. *Let $\delta > 0$ and let X and Y be quantum sets each equipped with δ-forms. Then it follows from [38, Theorem 4.7] that the quantum automorphism groups of the spaces X and Y are monoidally equivalent. In view of Theorem 4.2.11, this is equivalent to saying that the quantum complete graphs K_X and K_Y are **C***-quantum isomorphic. In particular,*

- *For each $n \geq 4$, we have $K_{n^2} \cong_{qc} K_{X_n}$, where K_{X_n} is the quantum complete graph associated to the quantum set $X_n = (M_n(\mathbb{C}), n^{-1}\mathit{Tr}(\cdot))$.*

- *Let $Q \in M_n(\mathbb{C})$ with $Q > 0$, $\mathit{Tr}(Q) = 1$, $\mathit{Tr}(Q^{-1}) = \delta^2 > 0$, and consider the quantum set $Y = (M_n(\mathbb{C}), \psi_Y = \mathit{Tr}(Q\cdot), \delta^2\psi_Y(\cdot)1)$. Then $K_Y \cong_{C^*} K_X$ for any quantum set X equipped with a δ-form.*

In particular, quantum isomorphic quantum graphs need not have the same dimension.

4.2.3 Applications to LBCS games

We are interested in considering the notion of equivalence for games. An extension of our result is shown in [22]. In order to do this, the key concept we need is the concept of a *hereditary $*$-algebra.*

Definition 4.2.14. *A $*$-algebra $\mathcal{A}$ is called* **hereditary** *provided that, whenever $n \in \mathbb{N}$ and $x_1, ..., x_n \in \mathcal{A}$ are such that $\sum_{i=1}^n x_i^* x_i = 0$, then $x_i = 0$ for all $1 \leq i \leq n$.*

One key advantage of hereditary $*$-algebras is that if $\mathcal{A}$ is a hereditary $*$-algebra and we set

$$\mathcal{P} = \left\{ x \in \mathcal{A} : \exists x_1, ..., x_n \in \mathcal{A} \text{ such that } x = \sum_{i=1}^n x_i^* x_i \right\},$$

then $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. Thus, we may define, for $a = a^*$ and $b = b^*$, a partial order by

$$a \leq b \iff \exists x_1, ..., x_n \in \mathcal{A} \text{ such that } b - a = \sum_i x_i^* x_i.$$

We note that if $a \leq b$ and $b \leq a$, then $a = b$.

Given a $*$-algebra $\mathcal{A}$, the smallest two-sided, $*$-closed hereditary ideal $\mathcal{I}$ containing 0 is called the *hereditary kernel* of $\mathcal{A}$, and the quotient $\mathcal{A}/\mathcal{I}$ is denoted by $\mathcal{A}_{hered}$. Given a synchronous game $\mathcal{G}$, we let $\mathcal{A}_{hered}(\mathcal{G})$ denote the hereditary quotient of $\mathcal{A}(\mathcal{G})$. Note that by Theorem 4.2.9, $\mathcal{A}(Iso(X,Y))$ admits a faithful tracial state whenever $\mathcal{A}(Iso(X,Y)) \neq 0$, and therefore $\mathcal{A}(Iso(X,Y)) = \mathcal{A}_{hered}(Iso(X,Y))$.

If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with $\mathcal{B}$ hereditary and $\pi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then the kernel of π contains the hereditary kernel of $\mathcal{A}$ and so induces a $*$-homomorphism $\tilde{\pi} : \mathcal{A}_{hered} \to \mathcal{B}$. Hence, for any pair of $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, every $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ induces a $*$-homomorphism $\tilde{\pi} : \mathcal{A}_{hered} \to \mathcal{B}_{hered}$.

Definition 4.2.15. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two synchronous games. We say that $\mathcal{G}_1$ and $\mathcal{G}_2$ are* **$*$-equivalent** *if there exist unital $*$-homomorphisms $\pi : \mathcal{A}(\mathcal{G}_1) \to \mathcal{A}(\mathcal{G}_2)$ and $\rho : \mathcal{A}(\mathcal{G}_2) \to \mathcal{A}(\mathcal{G}_1)$. We say that $\mathcal{G}_1$ and $\mathcal{G}_2$ are* **hereditarily $*$-equivalent** *if there exist unital $*$-homomorphisms $\pi : \mathcal{A}_{hered}(\mathcal{G}_1) \to \mathcal{A}_{hered}(\mathcal{G}_2)$ and $\rho : \mathcal{A}_{hered}(\mathcal{G}_2) \to \mathcal{A}_{hered}(\mathcal{G}_1)$.*

We allow the possibility that one of the two algebras is (0), in which case $1 = 0$ in that algebra. In this case, equivalence of the algebras implies that the other algebra is also (0).

Note that we do not require π and ρ to be mutual inverses or even one-to-one, just unital. The reason for examining this relation is given below.

Proposition 4.2.16. *Let $t \in \{loc, q, qa, qc, C^*\}$. If $\mathcal{G}_1, \mathcal{G}_2$ are synchronous games that are hereditarily $*$-equivalent, then $\mathcal{G}_1$ has a perfect t-strategy if and only if $\mathcal{G}_2$ has a perfect t-strategy. If, in addition, the games $\mathcal{G}_1$ and $\mathcal{G}_2$ are $*$-equivalent, then $\mathcal{G}_1$ has a perfect A^*-strategy if and only if $\mathcal{G}_2$ has a perfect A^*-strategy.*

Proof. We do the case $t = q$, the rest are similar. First assume that the algebras are $*$-equivalent. If $\mathcal{G}_2$ has a perfect q-strategy, then there is a unital $*$-morphism $\gamma : \mathcal{A}(\mathcal{G}_2) \to M_d$ for some d. Composing with π yields a $*$-homomorphism from $\mathcal{A}(\mathcal{G}_1)$ into M_d, and so, $\mathcal{G}_1$ has a perfect q-strategy. Since M_d is a hereditary $*$-algebra, the same reasoning applies when the algebras are hereditarily $*$-equivalent. The converse is clear, as are the remaining cases. $\square$

We now introduce another game which we will show is hereditarily $*$-equivalent to a graph isomorphism game.

We will now look at the connections between $Iso(X, Y)$ and $syncBCS(A, b)$. Let us recall from [2, Section 6] the graph $G_{A,b}$ defined for a linear system $Ax = b$ over $\mathbb{Z}_2$.

Definition 4.2.17. *Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}_2$ and $b \in \mathbb{Z}_2^n$. Define a graph $G_{A,b}$ with the following data:*

1. *the vertices of $G_{A,b}$ are pairs (i, x) where $i \in \{1, \ldots, m\}$ and $x \in S_i^b$;*
2. *there is an edge between distinct vertices (i, x) and (j, y) if and only if there exists some $k \in V_i \cap V_j$ for which $x_k \neq y_k$; that is, x and y are inconsistent solutions.*

We are now ready to state the main theorem of this section.

Theorem 4.2.18. *Let $A = (a_{i,j})$ be an $m \times n$ matrix over $\mathbb{Z}_2$ and let $b \in \mathbb{Z}_2^n$. Then the following three synchronous games:*

1. *$syncBCS(A, b)$,*
2. *$Iso(G_{A,b}, G_{A,0})$,*
3. *$Hom(K_m, \overline{G_{A,b}})$,*

are hereditarily $*$-equivalent.

Two examples of graphs which are quantum isomorphic but not classically isomorphic was shown in [2] utilizing this equivalence between the two games. In that example, both of the graphs had 24 vertices.

Combining Theorem 4.2.18 with Theorem 4.2.9 yields the following consequences.

Corollary 4.2.19. *Let $A = (a_{i,j})$ be an $m \times n$ matrix over $\mathbb{Z}_2$ and let $b \in \mathbb{Z}_2^n$. The following are equivalent:*

1. $\mathcal{A}_{hered}(syncBCS(A,b)) \neq (0)$*,*

2. $syncBCS(A,b)$ *has a perfect C*-strategy,*

3. $syncBCS(A,b)$ *has a perfect qc-strategy.*

The proof of Theorem 4.2.18 borrows some ideas from the proof of [26, Theorem 5.4].

Proof. We begin by constructing a unital $*$-homomorphism from $\mathcal{A}(Iso(G_{A,b}, G_{A,0}))$ to $\mathcal{A}(syncBCS(A,b))$. By our earlier remarks, this $*$-homomorphism will induce a unital $*$-homomorphism from

$\mathcal{A}_{hered}(Iso(G_{A,b}, G_{A,0}))$ to $\mathcal{A}_{hered}(syncBCS(A,b))$.

The algebra $\mathcal{A}(syncBCS(A,b))$ is generated by projections $e_{i,x}$ for $i = 1, \ldots, m$ and $x \in \mathbb{Z}_2^n$ satisfying $\sum_x e_{i,x} = 1$ for all i, $e_{i,x}e_{i,y} = 0$ if $x \neq y$. Moreover, given input i, if $x \notin S_i^b$, then they lose for all (j,y), from this it follows that $e_{i,x} = 0$ if $x \notin S_i^b$. Also, if $x \in S_i^b$ and $y \in S_j^b$, then $e_{i,x}e_{j,y} = 0$ if there is a $k \in V_i \cap V_j$ with $x_k \neq y_k$.

Let $S_i^0 \subseteq \mathbb{Z}_2^n$ denote the set of solutions to the ith equation of the linear system $Ax = 0$ and let $S_i^b \subseteq \mathbb{Z}_2^n$ denote the set of solutions to the ith equation of the linear system $Ax = b$. Note that if $y \in S_i^0$ and $x \in S_i^b$, then $x + y \in S_i^b$. Moreover, for $x \in S_i^b$, the map $S_i^0 \to S_i^b$ given by $y \mapsto x + y$ is a bijection.

The algebra $\mathcal{A}(Iso(G_{A,b}, G_{A,0}))$ is generated by projections $e_{(i,x),(j,y)}$ with $(i,x) \in V(G_{A,b})$ and $(j,y) \in V(G_{A,0})$, satisfying certain relations. For $(i,x) \in V(G_{A,b})$ and $(j,y) \in V(G_{A,0})$, define

$$q_{(i,x),(j,y)} = \begin{cases} e_{i,x+y} & i = j \\ 0 & i \neq j \end{cases}$$

and note that each $q_{(i,x),(j,y)}$ is a projection. For $(i,x) \in V(G_{A,b})$, we have

$$\sum_{(j,y) \in V(G_{A,0})} q_{(i,x),(j,y)} = \sum_{j=1}^{n} \sum_{y \in S_j^0} q_{(i,x),(j,y)} = \sum_{y \in S_i^0} e_{i,x+y} = \sum_{z \in S_i^b} e_{i,z} = 1.$$

A similar computation shows that for all $(j, y) \in V(G_{A,0})$, we have

$$\sum_{(i,x)\in V(G_{A,b})} q_{(i,x),(j,y)} = 1.$$

We need to show that for all $(i, x), (i', x') \in V(G_{A,b})$ and $(j, y), (j', y') \in V(G_{A,0})$, the implication

$$q_{(i,x),(j,y)}q_{(i',x'),(j',y')} \neq 0 \quad \Rightarrow \quad \mathrm{rel}((i, x), (i', x')) = \mathrm{rel}((j, y), (j', y'))$$

holds. To this end, suppose $q_{(i,x),(j,y)}q_{(i',x'),(j',y')} \neq 0$. Then $i = j$, $i' = j'$, and $e_{i,x+y}e_{i',x'+y'} \neq 0$. We consider several cases.

Suppose first $i = i'$. Then we have $x + y = x' + y'$. If $x = x'$, then $y = y'$ and we have both $(i, x) = (i', x')$ and $(j, y) = (j', y')$ so the right hand side of the implication holds in this case. Conversely, if $x \neq x'$ and $y \neq y'$, then $(i, x) \neq (i', x')$ and $(j, y) \neq (j', y')$. Note also that since $i = i'$, x and x' are necessarily inconsistent solutions so that (i, x) and (i', x') are adjacent. Similar reasoning shows (j, y) and (j', y') are adjacent. Hence the right hand side of the implication holds.

Now assume $i \neq i'$ so that, in particular, $(i, x) \neq (i', x')$. If (i, x) and (i', x') are adjacent, there is a $k \in V_i \cap V_{i'}$ such that $x_k \neq x'_k$. On the other hand, as $e_{i,x+y}e_{i',x'+y'} \neq 0$, we know $x_k + y_k = x'_k + y'_k$. Therefore, $y_k \neq y'_k$ so that (i, y) and (i', y') are adjacent. Finally, suppose (i, x) and (i', x') are not adjacent. Then $x_k = x'_k$ for all $i \in V_i \cap V_{i'}$. Again since $e_{i,x+y}e_{i',x'+y'} \neq 0$, we also know $x_k + y_k = x'_k + y'_k$ for all $k \in V_i \cap V_{i'}$ and therefore $y_k = y'_k$ for all $k \in V_i \cap V_{i'}$ so that (j, y) and (j', y') are not adjacent. This covers all cases.

Now, by the fact that $\mathcal{A}(Iso(G_{A,b}, G_{A,0}))$ is the universal $*$-algebra with projections satisfying these properties, we have that the map $e_{(i,x),(j,y)} \to q_{(i,x),(j,y)} \in \mathcal{A}(syncBCS(A, b))$ defines the desired unital $*$-homomorphism.

Now we prove that there is a unital $*$-homomorphism from $\mathcal{A}(Hom(K_m, \overline{G_{A,b}}))$ to $\mathcal{A}(Iso(G_{A,b}, G_{A,0}))$. Note that for any graph X we have that $\mathcal{A}(Hom(K_m, X))$ is generated by projections, $e_{i,x}$, $1 \leq i \leq m$, $x \in V(X)$ satisfying $\sum_x e_{i,x} = 1$, $e_{i,x}e_{i,y} = 0$, $x \neq y$ and $i \neq j, (x, y) \notin E(X) \implies e_{i,x}e_{j,y} = 0$. Since we are interested in $Hom(K_m, \overline{X})$, this last relation changes to $i \neq j, (x,y) \in E(X) \implies e_{i,x}e_{j,y} = 0$. For each $(i,x) \in V(G_{A,b})$ and $1 \leq j \leq m$ we define an element $p_{j,(i,x)} \in \mathcal{A}(Iso(G_{A,b}, G_{A,0}))$ by setting $p_{j,(i,x)} = e_{(i,x),(j,0)}$. We have that $\sum_{(i,x)\in V(G_{A,b})} p_{j,(i,x)} = 1$ and $p_{j,(i,x)}p_{j,(i',x')} = 0$ when $(i,x) \neq (i',x')$ by the magic permutation relations.

Finally, if $j \neq l$ and $((i,x),(i',x')) \in E(G_{A,b})$ then $rel((j,0),(l,0)) = +1$ while $rel((i,x),(i',x')) = -1$. Hence,

$$p_{j,(i,x)}p_{l,(i',x')} = e_{(i,x),(j,0)}e_{(i',x'),(l,0)} = 0.$$

This shows that the map from $\mathcal{A}(Hom(K_m, \overline{G_{A,b}}))$ to $\mathcal{A}(Iso(G_{A,b}, G_{A,0}))$ given by $e_{j,(i,x)} \to p_{j,(i,x)}$ defines a unital $*$-homomorphism and again this will induce a unital *-homomorphism between their hereditary quotients.

Finally, we must exhibit a unital $*$-homomorphism from $\mathcal{A}(syncBCS(A,b))$ into $\mathcal{A}_{hered}(Hom(K_m, \overline{G_{A,b}}))$.

This latter algebra is generated by projections $e_{i,(j,x)}$, $1 \leq i \leq m$, $(j,x) \in V(G_{A,b})$, i.e., $x \in S_j^b$. These satisfy $\sum_{j,x} e_{i,(j,x)} = 1$ for all i, and $e_{i,(j,x)}e_{i,(k,y)} = 0$ whenever $(j,x) \neq (k,y)$. Moreover, since (i,l) is an edge in K_m whenever $i \neq l$, we have that when $i \neq l$ and $((j,x),(k,y))$ is not an edge in $\overline{G_{A,b}}$ (meaning that $x \in S_j^b$ and $y \in S_k^b$ are inconsistent solutions), then $e_{i,(j,x)}e_{l,(k,y)} = 0$.

Note that if $x, y \in S_i^b$ and $x \neq y$, then $e_{k,(i,x)}e_{k,(i,y)} = 0$. If $k \neq j$, then k and j are connected by an edge in K_m, while (i,x) and (i,y) are not connected by an edge in $\overline{G_{A,b}}$, so that $e_{k,(i,x)}e_{j,(i,y)} = 0$. From these facts, it follows that

$$p_i := \sum_{k=1}^{m} \sum_{x \in S_i^b} e_{k,(i,x)}$$

is a self-adjoint idempotent. Set $q_i = 1 - p_i = q_i^2$. Then

$$\sum_{k=1}^{m} q_i^2 = \sum_{k=1}^{m} (1 - p_i) = m \cdot 1 - \sum_{k=1}^{m} \sum_{i=1}^{m} \sum_{x \in S_i^b} e_{k,(i,x)} = 0,$$

using the fact that $\sum_{j,x} e_{i,(j,x)} = 1$ for all i. Thus, we have that

$$q_i = 0 \text{ and } p_i = 1, \ \forall 1 \leq i \leq m.$$

For $x \in S_i^b$, set

$$f_{i,x} = \sum_{k=1}^{m} e_{k,(i,x)}.$$

Then $f_{i,x} = f_{i,x}^*$ and for $k \neq j$, we have that k, j are connected by an edge in K_m, while (i,x) is not connected to (i,x) by an edge; hence,

$$f_{i,x}^2 = \sum_{k,j=1}^{m} e_{k,(i,x)} e_{j,(i,x)} = \sum_{k=1}^{m} e_{k,(i,x)} = f_{i,x},$$

so that $f_{i,x}$ is a self-adjoint idempotent. Also, for $x, y \in S_i^b$ with $x \neq y$, we have that

$$f_{i,x} f_{i,y} = \sum_{j,k=1}^{m} e_{k,(i,x)} e_{j,(i,y)} = \sum_{k=1}^{m} e_{k,(i,x)} e_{k,(i,y)} = 0,$$

and

$$\sum_{x \in S_i^b} f_{i,x} = \sum_{k=1}^{m} \sum_{x \in S_i^b} e_{k,(i,x)} = p_i = 1.$$

Thus, for each i, $\{f_{i,x} : x \in S_i^b\}$ is a set of self-adjoint idempotents whose sum is 1.

Finally, if (i,x) and (j,y) are inconsistent solutions, then

$$f_{i,x} f_{j,y} = \sum_{k,h=1}^{m} e_{k,(i,x)} e_{h,(j,y)}.$$

When $h = k$, each of these products is 0. For $h \neq k$, we have that h and k are connected by an edge in K_m and so the product will be 0, since x and y being inconsistent solutions implies that (i,x) and (j,y) are not connected by an edge in $\overline{G_{A,b}}$.

Thus, the set $\{f_{i,x}\}$ satisfies the relations on the generators of the free algebra $\mathcal{A}(syncBCS(A,b))$ and they induce a unital $*$-homomorphism from $\mathcal{A}(syncBCS(A,b))$ into

$\mathcal{A}_{hered}(Hom(K_m, \overline{G_{A,b}}))$, from which the result follows. □

4.3 Bigalois extensions and quantum isometries of metric spaces

In this section, we explore the quantum isometries between two W^*-quantum metric spaces. We restrict our attention in this section to the finite dimensional case, where interesting universal algebras are guaranteed to exist. This section uses techniques from [10].

Definition 4.3.1. *Consider two finite dimensional quantum metric spaces $X = (\mathcal{M}_1, \mathcal{H}_1, \mathcal{V}_t)$ and $Y = (\mathcal{M}_2, \mathcal{H}_2, \mathcal{W}_t)$ where the canonical traces on $\mathcal{M}_1$ and $\mathcal{M}_2$ are fixed and $\{e_j\}$ and $\{f_k\}$ are orthonormal bases for $\mathcal{M}_1$ and $\mathcal{M}_2$. Moreover, by Theorem 3.3.9 we may assume $\mathcal{H}_i = L^2(\mathcal{M}_i)$.*

We define $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ to be the universal C^*-algebra generated by the coefficients of a unitary $P = [p_{ij}] \in \mathcal{O}(G^{\mathcal{V},\mathcal{W}}) \otimes B(\mathcal{H}_1, \mathcal{H}_2)$ with relations giving a unital $*$-homomorphism

$$\begin{aligned} \delta_{\mathcal{V},\mathcal{W}} : \mathcal{M}_1 &\to \mathcal{M}_2 \otimes \mathcal{O}(G^{\mathcal{V},\mathcal{W}}) \\ e_j &\mapsto \sum_k f_k \otimes p_{kj} \end{aligned}$$

and ensuring the conjugation map given by

$$\begin{aligned} \alpha_{\mathcal{V},\mathcal{W}} : B(\mathcal{H}_1) &\to B(\mathcal{H}_2) \otimes \mathcal{O}(G^{\mathcal{V},\mathcal{W}}) \\ T &\mapsto P(T \otimes 1)P^* \end{aligned}$$

satisfies $\alpha_{\mathcal{V},\mathcal{W}}(\mathcal{V}_t) \subseteq \mathcal{W}_t \otimes \mathcal{O}(G^{\mathcal{V},\mathcal{W}})$.

This definition satisfies two crucial criteria:

1. $G^{\mathcal{V},\mathcal{V}} = G^{\mathcal{V}}$, as desired
2. For classical metric spaces $(X, d_X), (Y, d_Y)$ and their corresponding W^*-quantum metric spaces $(\ell^2(X), \mathcal{V}_t)$ and $(\ell^2(Y), \mathcal{W}_t)$, we have $\mathcal{O}(G^{\mathcal{V},\mathcal{W}}) = \mathcal{A}(Isom(X, Y))$

Proposition 4.3.2. *Given two quantum graphs $X_1 = (\mathcal{B}_1, \psi_1, A_1)$ and $X_2 = (\mathcal{B}_2, \psi_2, A_2)$, we let $(\mathcal{M}, \mathcal{V}_t)$ and $(\mathcal{N}, \mathcal{W}_t)$ be the associated W^*-metric spaces.*

If the two quantum graphs are quantum isometric, then so are their associated W^*-metric spaces.

Proof. If the quantum graphs X_1 and X_2 are quantum isomorphic, then there exists some C^*-algebra $\mathcal{C}$ and some unitary $P \in \mathcal{C} \otimes B(L^2(X_1), L^2(X_2))$ which intertwines the unit maps $\eta_{\mathcal{B}_i}$ and the multiplication maps $m_{\mathcal{B}_i}$ such that the map

$$\alpha_{12} : B(L^2(\mathcal{B}_1)) \to B(L^2(\mathcal{B}_2)) \otimes \mathcal{C}$$

$$T \mapsto P(T \otimes 1)P^*$$

satisfies $P(A_1 \otimes 1) = (A_2 \otimes 1)P$.

For the associated operator systems $\mathcal{S}_i = \mathcal{V}_i$ defined in Example 3.2.4, one can show that $\alpha_{12}(\mathcal{S}_1) \subseteq \mathcal{S}_2 \otimes \mathcal{C}$. It then immediately follows that $\alpha_{12}(\mathcal{S}_1^k) \subseteq \mathcal{S}_2^k \otimes \mathcal{C}$. □

Definition 4.3.3. *We say that the quantum metric spaces $(\mathcal{M}_1, \mathcal{H}_1, \mathcal{V}_t)$ and $(\mathcal{M}_2, \mathcal{H}_2, \mathcal{W}_t)$ are A^* quantum isometric if $\mathcal{O}(G^{\mathcal{V},\mathcal{W}}) \neq 0$, and we write $(\mathcal{M}_1, \mathcal{V}_t) \cong_{A^*} (\mathcal{M}_2, \mathcal{W}_t)$. We say they are are C^* quantum isometric if $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ has a C^*-representation, and we write $(\mathcal{M}_1, \mathcal{V}_t) \cong_{C^*} (\mathcal{M}_2, \mathcal{W}_t)$. Finally, we say they are qc-quantum isometric if $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ admits a tracial state and write $(\mathcal{M}_1, \mathcal{V}_t) \cong_{qc} (\mathcal{M}_2, \mathcal{W}_t)$, so that by the work of [30] this is the same as the existing notation for classical metric spaces.*

Our next goal is to show that $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ admits a natural structure as an $\mathcal{O}(G^{\mathcal{V}}) - \mathcal{O}(G^{\mathcal{W}})$ bigalois extension.

Theorem 4.3.4. *If $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ is non-zero, then there exists a $\mathcal{O}(G^{\mathcal{V}}) - \mathcal{O}(G^{\mathcal{W}})$ bigalois extension.*

Proof. We show that $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ is an $\mathcal{O}(G^{\mathcal{W}}) - \mathcal{O}(G^{\mathcal{V}})$ bicomodule, following the proof in Section 4.2.

We define "cocomposition" $*$-morphisms

$$\gamma_{\mathcal{W}} : G^{\mathcal{W}} \to G^{\mathcal{V},\mathcal{W}} \otimes G^{\mathcal{W},\mathcal{V}} \qquad\qquad \gamma_{\mathcal{W}}(u_{ij}) = \sum_k p_{ik} \otimes q_{kj}$$

$$\gamma_{\mathcal{V}} : G^{\mathcal{V}} \to G^{\mathcal{W},\mathcal{V}} \otimes G^{\mathcal{V},\mathcal{W}} \qquad\qquad \gamma_{\mathcal{V}}(u_{ij}) = \sum_k q_{ik} \otimes p_{kj}$$

where $q = [q_{ij}]$ is the matrix of generators of $G^{\mathcal{W},\mathcal{V}}$.

We now have a two-object cocategory $\mathcal{C}$: the four algebras $G^{\mathcal{V}}, G^{\mathcal{W}}, G^{\mathcal{V},\mathcal{W}}$ and $G^{\mathcal{W},\mathcal{V}}$ are thought of as dual to "spaces of morphisms" between two objects ($x \mapsto x$ for $G^{\mathcal{V}}$, $x \mapsto y$ for $G^{\mathcal{V},\mathcal{W}}$, etc) and the γ maps are dual to morphism composition.

Next, we make $\mathcal{C}$ into a cogroupoid by defining coinversion maps

$$S_{\mathcal{V},\mathcal{W}} : G^{\mathcal{W},\mathcal{V}} \to G^{\mathcal{V},\mathcal{W}}$$

$$S_{\mathcal{W},\mathcal{V}} : G^{\mathcal{V},\mathcal{W}} \to G^{\mathcal{W},\mathcal{V}}$$

Let $F = F_{\mathcal{V}} \in M_n$ and $G = F_{\mathcal{W}} \in M_m$ be matrices with the property that $Fe_i = e_i^*$ and similarly for G, so that $\overline{F} = F^{-1}$ and $\overline{G} = G^{-1}$. Then we have the following involutivity of morphisms on the left equivalent to the equalities on the right

$$\begin{aligned} \delta_{\mathcal{V}} &: \mathcal{H}_1 \to \mathcal{H}_1 \otimes G^{\mathcal{V}} & (1 \otimes F)\overline{u} &= u(1 \otimes F) \\ \delta_{\mathcal{W}} &: \mathcal{H}_2 \to \mathcal{H}_2 \otimes G^{\mathcal{W}} & (1 \otimes G)\overline{v} &= v(1 \otimes G) \\ \delta_{\mathcal{V},\mathcal{W}} &: \mathcal{H}_1 \to \mathcal{H}_2 \otimes G^{\mathcal{V},\mathcal{W}} & (1 \otimes G)\overline{p} &= p(1 \otimes F) \end{aligned} \tag{4.10}$$

For ease of notation we will start writing uF for $u(1 \otimes F)$. Then we have $G^{-1}pF = \overline{p}$ and $F^{-1}qG = \overline{q}$.

Then we can check that we have a unital algebra homomorphism

$$\mathcal{H}_1 \to \mathcal{H}_2 \otimes (G^{\mathcal{V},\mathcal{W}})^{op}$$
$$f_i \mapsto \sum_j e_j \otimes p_{ij}^*$$

Applying G to both sides and noting that $e_j = FF^{-1}e_j$ then the map above is involutive with respect to the modified $*$-structure $\star$ on $(G^{\mathcal{V},\mathcal{W}})^{op}$ given by $(p^*)^\star = (F^{-1}p^*F)^t$.

The universal property of $G^{\mathcal{V},\mathcal{W}}$ implies that the homomorphism above factors as $(\mathrm{id} \otimes S_{\mathcal{V},\mathcal{W}})\delta_{\mathcal{V}}$ where $S_{\mathcal{V},\mathcal{W}}$ is a conjugate-linear anti-morphism defined by

$$S_{\mathcal{V},\mathcal{W}} : G^{\mathcal{W},\mathcal{V}} \to G^{\mathcal{V},\mathcal{W}} \qquad q \mapsto p^* \quad q^* \mapsto G^t\overline{p}F^{-t}$$
$$S_{\mathcal{W},\mathcal{V}} : G^{\mathcal{V},\mathcal{W}} \to G^{\mathcal{W},\mathcal{V}} \qquad p \mapsto q^* \quad p^* \mapsto F^t\overline{p}G^{-t}$$

Since we have shown that $\mathcal{C}$ is a connected cogroupoid then if $G^{\mathcal{V},\mathcal{W}}$ is non-zero, [8] and [10] shows that $G^{\mathcal{V},\mathcal{W}}$ is a $G^{\mathcal{V}} - G^{\mathcal{W}}$ bigalois extension. □

Utilizing the work in [7], we can show the existence of a bi-invariant state ω for the $G^{\mathcal{V}} - G^{\mathcal{W}}$ bigalois extension, referenced in Theorem 4.1.6.

Consider $n \in \mathbb{N}$ and matrices $F_i \in GL_n(\mathbb{C})$. Define $\mathcal{O}(U_{F_1}^+, U_{F_2}^+)$ to be the unital $*$-algebra generated by the coefficients z_{ij} of the $n_1 \times n_2$ matrix $z = [z_{ij}] \in M_{n_1,n_2}(\mathcal{O}(U_{F_1}^+, U_{F_2}^+))$ the relations that z and $F_1\overline{z}F_2^{-1}$ are unitary where $\overline{z} = [z_{ij}^*]$. Note that if $\mathcal{O}(U_{F_1}^+, U_{F_2}^+) \neq 0$ then $\mathcal{O}(U_{F_1}^+, U_{F_2}^+)$ is a $\mathcal{O}(U_{F_1}^+) - \mathcal{O}(U_{F_2}^+)$ bigalois extension with respect to the bicomodule structure given by

$$\alpha_{F_1,F_2} : \mathcal{O}(U^+_{F_1}, U^+_{F_2}) \to \mathcal{O}(U^+_{F_1}) \otimes \mathcal{O}(U^+_{F_1}, U^+_{F_2}) \qquad \alpha_{F_1,F_2}(z_{ij}) = \sum_{k=1}^{n_1} u_{ik} \otimes z_{kj}$$

$$\beta_{F_1,F_2} : \mathcal{O}(U^+_{F_1}, U^+_{F_2}) \to \mathcal{O}(U^+_{F_1}, U^+_{F_2}) \otimes \mathcal{O}(U^+_{F_2}) \qquad \beta_{F_1,F_2}(z_{ij}) = \sum_{\ell=1}^{n_2} z_{i\ell} \otimes v_{\ell j}$$

where $u = [u_{ij}]$ is the fundamental representation of $U^+_{F_1}$ and $v = [v_{ij}]$ is the fundamental representation of $U^+_{F_2}$.

Theorem 4.3.5. *([7]) Let G be a compact quantum group and (Z, α) a left $\mathcal{O}(G)$-Galois extension. Let $F \in GL_n(\mathbb{C})$ be such that $G < U^+_F$ with corresponding surjective morphism $\pi : \mathcal{O}(U^+_F) \to O(G)$. If there exists $F_1 \in GL_{n_1}(\mathbb{C})$ and a surjective $*$-homomorphism $\sigma : \mathcal{O}(U^+_F, U^+_{F_1}) \to Z$ satisfying $\alpha \circ \sigma = (\pi \otimes \sigma)\alpha_{F,F_1}$ then Z admits a left-invariant state $\omega : Z \to \mathbb{C}$.*

Thus, we have the following theorem:

Theorem 4.3.6. *Let $(\mathcal{M}_1, \mathcal{V}_t)$ and $(\mathcal{M}_2, \mathcal{W}_t)$ be finite quantum metric spaces. If $G^{\mathcal{V},\mathcal{W}} \neq 0$, then there exists a faithful, bi-invariant, tracial state $\omega : G^{\mathcal{V},\mathcal{W}} \to \mathbb{C}$.*

Proof. Utilizing the proof of Theorem 4.3.4, we may consider the matrices $F_1 = F_{\mathcal{V}}$ and $F_2 = F_{\mathcal{W}}$. Equation (4.10) shows that we have surjective $*$-homomorphisms $\pi : \mathcal{O}(U^+_{F_{\mathcal{V}}}) \to G^{\mathcal{V}}$ and $\sigma : \mathcal{O}(U^+_{F_Y}, U^+_{F_X}) \to G^{\mathcal{V},\mathcal{W}}$ satisfying

$$\alpha \circ \sigma = (\pi \otimes \sigma)\alpha_{F_Y,F_X}.$$

Then by Theorem 4.1.5, $G^{\mathcal{V},\mathcal{W}}$ admits a $G^{\mathcal{W}} - G^{\mathcal{V}}$ invariant state, and it is tracial if and only if both $G^{\mathcal{V}}$ and $G^{\mathcal{W}}$ are of Kac type. □

Corollary 4.3.7. *Let $(\mathcal{M}_1, \mathcal{V}_t)$ and $(\mathcal{M}_2, \mathcal{W}_t)$ be finite dimensional quantum metric spaces. If $G^{\mathcal{V},\mathcal{W}}$ is non-zero, then the compact quantum groups $G^{\mathcal{V}}$ and $G^{\mathcal{W}}$ are monoidally equivalent, $G^{\mathcal{V}} \sim^{mon} G^{\mathcal{W}}$.*

Proof. This is a corollary of Theorem 4.3.6. By Theorem 4.1.5, $G^{\mathcal{V}}$ and $G^{\mathcal{W}}$ are monoidally equivalent. $\square$

Corollary 4.3.8. *Let $(\mathcal{M}_1, \mathcal{V}_t)$ and $(\mathcal{M}_2, \mathcal{W}_t)$ be finite dimensional quantum metric spaces. Then the following are equivalent:*

1. $(\mathcal{M}_1, \mathcal{V}_t) \cong_{A^*} (\mathcal{M}_2, \mathcal{W}_t)$
2. $(\mathcal{M}_1, \mathcal{V}_t) \cong_{C^*} (\mathcal{M}_2, \mathcal{W}_t)$
3. $(\mathcal{M}_1, \mathcal{V}_t) \cong_{qc} (\mathcal{M}_2, \mathcal{W}_t)$

Remark 4.3.9. *This theorem says that as soon as $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ is non-zero, then $\mathcal{O}(G^{\mathcal{V},\mathcal{W}})$ admits a tracial state. This is non-trivial and it has been shown that this phenomena is not true for other games. In [12], a graph homomorphism from a quantum graph to a classical graph was defined and studied. They showed that the game $*$-algebra is always non-zero when the output graph is K_4.*

By restricting our attention to classical metric spaces, we get one of the main results of the paper:

Corollary 4.3.10. *Let (X, d_X) and (Y, d_Y) be classical metric spaces. Then the following are equivalent:*

1. $X \cong_{A^*} Y$
2. $X \cong_{C^*} Y$
3. $X \cong_{qc} Y$

Proof. This is an immediate consequence of Corollary 4.3.8. $\square$

Remark 4.3.11. *An example of two classical graphs which are quantum isomorphic but not classically isomorphic is shown in [2]. Each of the graphs have 24 vertices, and naturally give rise to classical metric spaces of size 24 which are quantum isometric but not classically isometric.*

5. SUMMARY

We have proved that given two quantum objects (either quantum graphs or quantum metric spaces), if the quantum symmetry group between the two quantum objects is non-zero then it is immediately true that the two quantum symmetry groups of the two quantum objects are monoidally equivalent (Theorem 4.2.7 and Corollary 4.3.7). When restricted to classical objects, this says that the two classical objects are algebraically quantum isometric if and only if the corresponding game has a perfect quantum-commuting (qc)-strategy (Theorem 4.2.9 and Corollary 4.3.10).

One might consider the graph isomorphism game played with a quantum graph X and let G_X be its quantum automorphism group. Then for any compact quantum group G monoidally equivalent to G_X, one can construct from this monoidal equivalence a quantum graph Y, an isomorphism of quantum groups $G \cong G_Y$, and an algebraic quantum isomorphism $X \cong_{A^*} Y$ (Theorem 4.2.11).

We believe this should also be true for W^*-quantum metric spaces.

Conjecture 5.0.1. *Let $X = (\mathcal{M}_1, \mathcal{V}_t)$ be a W^*-quantum metric space and $G^{\mathcal{V}}$ be its quantum isometry group. Let G be another compact quantum group that is monoidally equivalent to $G^{\mathcal{V}}$. Then there exists a W^*-quantum metric space $Y = (\mathcal{M}_2, \mathcal{W}_t)$ so that $G = G^{\mathcal{W}}$, and we have a quantum isometry $X \cong_{A^*} Y$.*

There are a number of open questions regarding the scope of graph isomorphisms games, and this may be interesting for future research. We wonder what is the computational power of the class of graph isomorphism games? While it's likely that the class of graph isomorphism games are not enough to disprove Connes' embedding conjecture, we still wonder whether it is possible to produce a graph G whose quantum automorphism group is not hyperlinear; that is, for which $L^\infty(G)$ does not embed into an ultrapower of the hyperfinite II_1-factor R^ω.

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